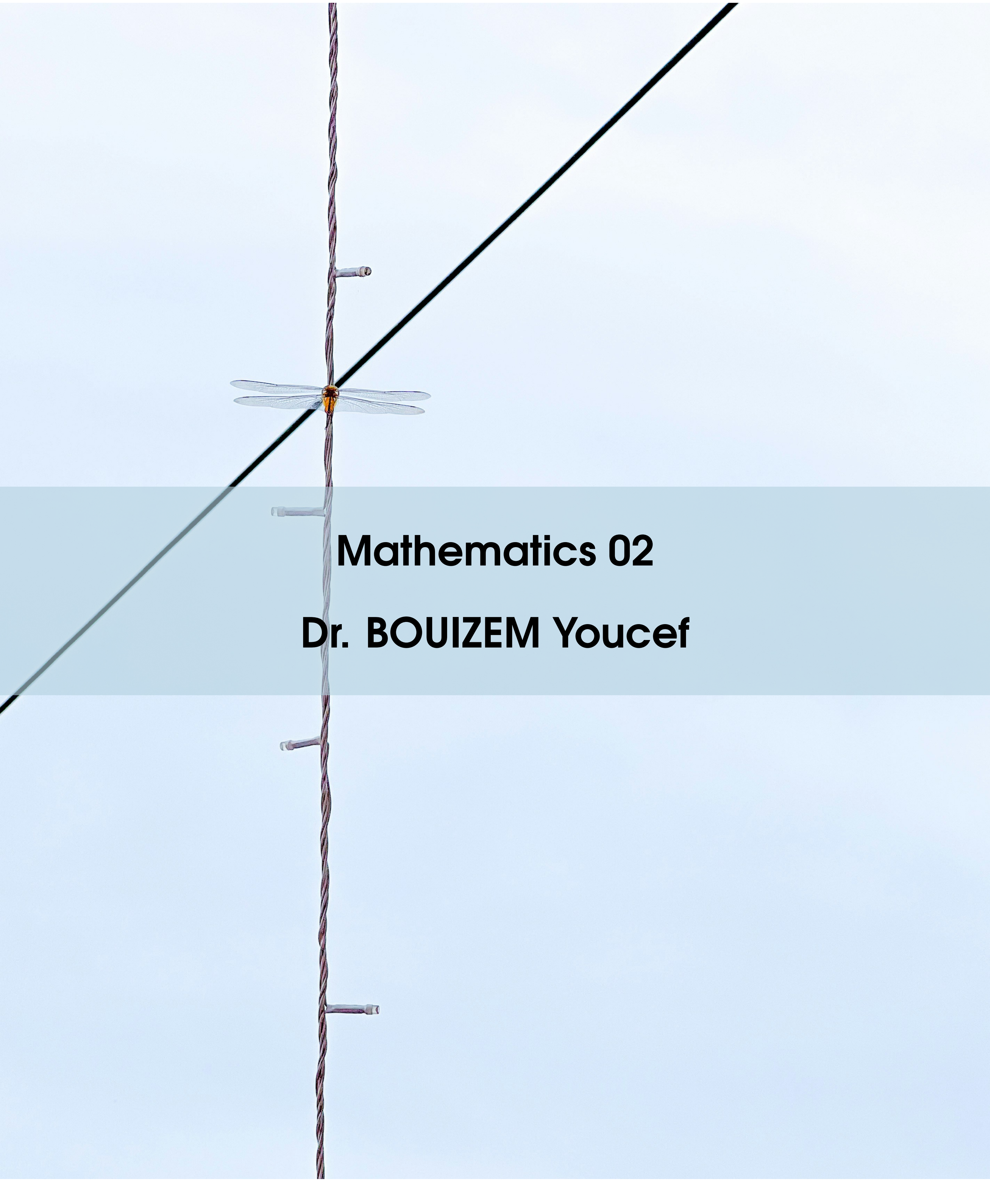

Mathematics 02

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Mathematics 02

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Distribution of Mathematics 2 Program

Program	Mathematics 2
Teaching Units	Fundamental
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Total Class Hours	67 hours
Assessment Method	40% Continuous Assessment - 60% Final Exam

Course Canvas

Chapter Title	**Contents**	**Weeks**
Matrices and determinants	Matrices (Definition, operations) Matrix associated to a linear map A linear map associated to a matrix Change of basis	02
Systems of linear equations	General concepts Study of the solution set Solving methods (Cramer, inverse, Gauss)	03
Integrals	Indefinite integrals Definite Integrals Integration of polynomial, rational and trigonometric functions	04
Differential equations	First order linear DE Second order linear DE with constant coefficients	03
Multi-variable functions	Limits, continuity and partial differentiation Differentiability Double and triple integrals	03

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Mathematics is a fundamental tool for understanding and solving problems in various fields, from the natural sciences to engineering and social sciences. One of its core branches is linear algebra, which deals with the study of vectors, matrices, and systems of linear equations. Another essential branch is calculus, which focuses on the study of rates of change and accumulation.

This textbook provides a comprehensive introduction to these two fundamental areas of mathematics. We will explore the concepts of matrices, systems of linear equations, integrals, differential equations and functions of several variables. These topics are interconnected and have wide-ranging applications in various fields.

Matrices are rectangular arrays of numbers or symbols arranged in rows and columns. They are used to represent data, solve systems of equations, and perform transformations in geometry. Systems of linear equations are sets of equations involving linear expressions. Solving these systems is a common task in mathematics and has practical applications in fields such as engineering, economics, and computer graphics.

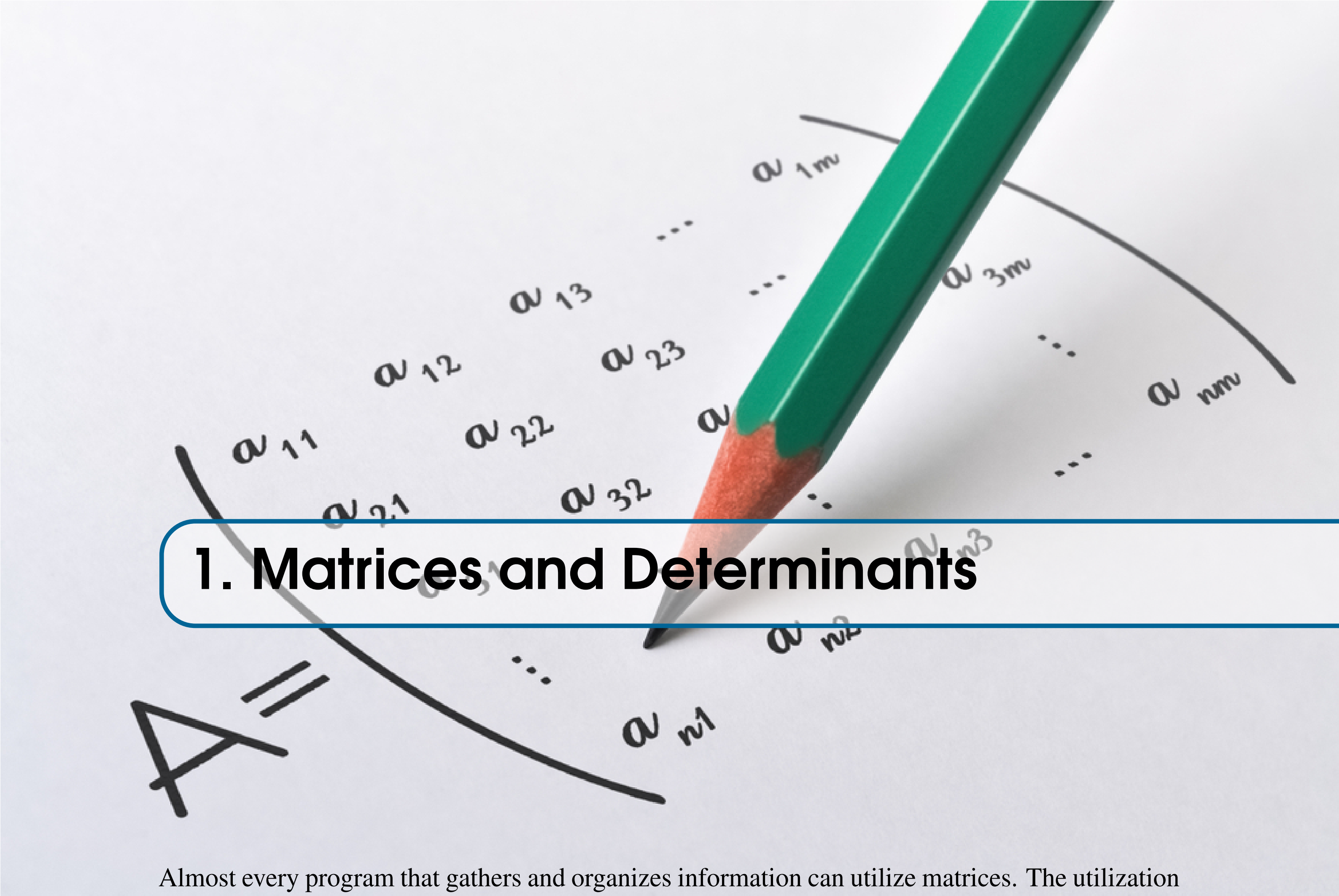
Integrals are used to calculate areas, volumes, and other quantities. They are a fundamental concept in calculus and have applications in physics, engineering, and economics. Differential equations are equations involving derivatives of functions. They are used to model various phenomena in physics, engineering, biology, and other fields.

This textbook provides a solid foundation in essential mathematical concepts that are crucial for first-year university students, since it covers the most common and important chapters (Matrices, systems, integrals and differential equations) about the unit Mathematics 2.

Throughout this document, the narrative is designed to captivate the reader's focus. Examples and solved applications are seamlessly integrated into the text, creating an immersive learning experience. Visual aids and illustrative examples are strategically placed to reinforce comprehension and spark curiosity. This modest textbook is very concise with little amount of proofs and easy problems. Therefore, I would recommend it as a first reading experience before getting into more detailed and elaborated sources.

In concluding this introduction, i would give many Thanks to **Dr. Anes MOULAI-KHATIR** for all his support and his serious contribution to this work.

Contents



1. Matrices and Determinants

Almost every program that gathers and organizes information can utilize matrices. The utilization of matrices has expanded with the rise in data availability across various aspects of life and business. They play a crucial role in arranging data across various scientific disciplines such as physics, chemistry, biology, genetics, computer science, meteorology and economics. They are also used for bringing images to life in movies and video games through animation.

1.1 Matrices

Matrices are tables of numbers. Most linear algebra problems are solved by manipulating matrices. this is true, particularly when solving linear systems.

In this chapter, $\mathbb{K}$ represents a field. We can think of $\mathbb{R}$ or $\mathbb{C}$.

1.1.1 Definitions and basic concepts

- a **matrix** A is a rectangular table of elements of $\mathbb{K}$.
- It is said to be of size $n \times p$ if the table possess n lines and p columns.
- The numbers of the table are called **coefficients** of A.
- the coefficient situated at i^{th} line and the j^{th} column is noted $a_{i,j}$.

This table is represented by:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,j} & \dots & a_{1,p} \\ a_{2,1} & a_{2,2} & \dots & a_{2,j} & \dots & a_{2,p} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i,1} & a_{i,2} & \dots & a_{i,j} & \dots & a_{i,p} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,j} & \dots & a_{n,p} \end{pmatrix} \quad \text{or} \quad A = \left(a_{i,j}\right)_{\substack{1\le i\le n \\ 1\le j\le p}}.$$

■ **Example 1.1**

$$A = \begin{pmatrix} 1 & -2 & 5 \\ 0 & 3 & 7 \end{pmatrix}$$

is a matrix 2×3 with, for example, $a_{1,1} = 1$ and $a_{2,3} = 7$. ■

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- Two matrices are **equal** when they have the same size and their corresponding coefficients are equal.
- The set of matrices with n line and p column and with coefficients in $\mathbb{K}$ are written $M_{n,p}(\mathbb{K})$. The elements of $M_{n,p}(\mathbb{R})$ are called **real matrices**.

- If $n = p$ (same number of lines and columns), the matrix is called **square matrix**. We write $M_n(\mathbb{K})$ instead of $M_{n,n}(\mathbb{K})$.

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{pmatrix}$$

The elements $a_{1,1}, a_{2,2}, \dots, a_{n,n}$ form the **principal diagonal** of the matrix.
- A matrix with only one line ($n = 1$) is called **line matrix** or **line vector**. We write

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,p} \end{pmatrix}.$$

- Similarly, A matrix with only one column ($p = 1$) is called **column matrix** or **column vector**. We write

$$A = \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{n,1} \end{pmatrix}.$$

- A matrix (of size $n \times p$) with all coefficients equal to zero, is called **null matrix** and is noted $0_{n,p}$ or simply 0. In the matrix calculus, the null matrix plays the role of the number 0 for the reals.

1.1.2 Addition of matrices

Definition 1.1.1 Let A and B two matrices of the same size $n \times p$. Their **sum** $C = A + B$ is the matrix of size $n \times p$ defined by

$$c_{ij} = a_{ij} + b_{ij}.$$

Example 1.2

$$\text{If} \qquad A = \begin{pmatrix} 3 & -2 \\ 1 & 7 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & 5 \\ 2 & -1 \end{pmatrix}$$

then

$$A + B = \begin{pmatrix} 3 & 3 \\ 3 & 6 \end{pmatrix}.$$

$$\text{however if} \qquad B' = \begin{pmatrix} -2 \\ 8 \end{pmatrix} \qquad \text{then} \quad A + B' \quad \text{is not defined}$$

■

Definition 1.1.2 The product of a matrix $A = (a_{ij})$ of $M_{n,p}(\mathbb{K})$ by a scalar $\alpha \in \mathbb{K}$ is the matrix (αa_{ij}) formed by multiplying each coefficient of A by α. It is noted $\alpha \cdot A$ (or simply αA).

Example 1.3

$$\text{If} \qquad A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \alpha = 2 \qquad \text{then} \qquad \alpha A = \begin{pmatrix} 2 & 4 & 6 \\ 0 & 2 & 0 \end{pmatrix}.$$

■

The matrix $(-1)A$ is the **opposite** of A and is written $-A$.
The **difference** $A - B$ is defined by $A + (-B)$.

Example 1.4

$$\text{If} \quad A = \begin{pmatrix} 2 & -1 & 0 \\ 4 & -5 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 4 & 2 \\ 7 & -5 & 3 \end{pmatrix}$$

then

$$A - B = \begin{pmatrix} 3 & -5 & -2 \\ -3 & 0 & -1 \end{pmatrix}.$$

■

Proposition 1.1.1 Let A, B and C three matrices belonging to $M_{n,p}(\mathbb{K})$. Let $\alpha \in \mathbb{K}$ and $\beta \in \mathbb{K}$ two scalars.

1. $A+B=B+A$: The sum is commutative,
2. $A+(B+C)=(A+B)+C$: The sum is associative,
3. $A+0=A$: the null matrix is the neutral element of the addition,
4. $(\alpha+\beta)A=\alpha A+\beta A$,
5. $\alpha(A+B)=\alpha A+\alpha B$.

1.1.3 Product of matrices

The product AB of two matrices A and B is defined if, and only if the number of columns of A is equal to number of lines of B.

Definition 1.1.3 — Product of two matrices. Let $A=(a_{ij})$ a matrix $n\times p$ and $B=(b_{ij})$ a matrix $p\times q$. Then the product $C=AB$ is a matrix $n\times q$ whose coefficients c_{ij} are defined by :

$$\boxed{c_{ij}=\sum_{k=1}^{p} a_{ik}b_{kj}=a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{ik}b_{kj}+\cdots+a_{ip}b_{pj}.}$$

It is common to dispose the calculation in the following manner.

$$\begin{array}{ccc} & \begin{pmatrix} & & \times & & \\ & & \times & & \\ & & \times & & \\ & & \times & & \end{pmatrix} & \leftarrow B \\ A\rightarrow \begin{pmatrix} \\ \\ \times & \times & \times & \times \\ \end{pmatrix} & \begin{pmatrix} & & | & & \\ & & | & & \\ - & - & - & c_{ij} & \\ & & & & \end{pmatrix} & \leftarrow AB \end{array}$$

Let :

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix} \qquad B=\begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 1 & 1 \end{pmatrix}$$

First test :

$$\begin{array}{cc} & \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix} & \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \end{array} \qquad \begin{array}{cc} & \begin{pmatrix} \color{red}{1} & 2 \\ \color{red}{-1} & 1 \\ \color{red}{1} & 1 \end{pmatrix} \\ \begin{pmatrix} \color{red}{1} & \color{red}{2} & \color{red}{3} \\ 2 & 3 & 4 \end{pmatrix} & \begin{pmatrix} \color{blue}{2} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \end{array}$$

Hence :

$$\begin{pmatrix} 2 & 7 \\ 3 & 11 \end{pmatrix} = AB$$

Proposition 1.1.2 1. The product of matrices is not commutative in general,

2. $AB = 0$ does not imply $A = 0$ or $B = 0$,
3. $AB = AC$ does not imply $B = C$,
4. $A(BC) = (AB)C$: associativity of the product,
5. $A(B+C) = AB+AC$ and $(B+C)A = BA+CA$: distributivity of the product compared to the sum.
6. $A \cdot 0 = 0$ and $0 \cdot A = 0$.

Definition 1.1.4 — Identity Matrix. The following square matrix is called the **identity matrix**:

$$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

Proposition 1.1.3 If A is a matrix $n \times p$, then

$$I_n \cdot A = A \qquad \text{and} \qquad A \cdot I_p = A.$$

Definition 1.1.5 — Invertible Matrices. Let $A \in M_n(\mathbb{K})$, we say that A is invertible if

$$\exists B \in M_n(\mathbb{K}), AB = I_n \text{ or } \exists C \in M_n(\mathbb{K}), CA = I_n$$

The inverse matrix of A is denoted A^{-1} and verifies $AA^{-1} = A^{-1}A = I_n$.

In the set $M_n(\mathbb{K})$ of square matrices of size $n \times n$ with coefficients in $\mathbb{K}$, the product of matrices is an internal operation : if $A, B \in M_n(\mathbb{K})$ then $AB \in M_n(\mathbb{K})$.

Particularly, we can multiply a square matrix by itself: we note $A^2 = A \times A$, $A^3 = A \times A \times A$.

Hence, we can define successive powers of matrices :

Definition 1.1.6 For all $A \in M_n(\mathbb{K})$, we define the successive power of matrices of A by $A^0 = I_n$ and $A^{p+1} = A^p \times A$ for all $p \in \mathbb{N}$. In other words,

$$A^p = \underbrace{A \times A \times \cdots \times A}_{p \text{ times}}.$$

We seek to calculate A^p with $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. we calculate A^2, A^3 and A^4 and we get :

$$A^2 = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \quad A^3 = A^2 \times A = \begin{pmatrix} 1 & 0 & 7 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{pmatrix} \quad A^4 = \begin{pmatrix} 1 & 0 & 15 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{pmatrix}.$$

A close look at the first powers allows one to think that the formula is: $A^p = \begin{pmatrix} 1 & 0 & 2^p - 1 \\ 0 & (-1)^p & 0 \\ 0 & 0 & 2^p \end{pmatrix}$.

Exercise : Prove the result by induction.

Let A a matrix of size $n \times n$. We say that A is a **lower triangular matrix** or **left triangular matrix** if its elements under the diagonal are nulls, in other words :

$$i < j \implies a_{ij} = 0.$$

A lower triangular matrix has the following form:

$$\begin{pmatrix} a_{11} & 0 & \cdots & \cdots & 0 \\ a_{21} & a_{22} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{pmatrix}$$

We say that A is an **upper triangular matrix** or **right triangular matrix.** if its elements over the diagonal are nulls, in other words:

$$i > j \implies a_{ij} = 0.$$

An upper triangular matrix has the following form:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & \cdots & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & a_{nn} \end{pmatrix}$$

Two lower triangular matrices (left and center), one upper triangular matrix (right):

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 3 & -2 & 3 \end{pmatrix} \qquad \begin{pmatrix} 5 & 0 \\ 1 & -2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

A matrix that is lower triangular **and** upper triangular is said to be **diagonal**. In other words: $i \neq j \implies a_{ij} = 0$.

■ **Example 1.5** If D is a diagonal matrix, we can easily calculate its powers D^p (by induction in p) :

$$D = \begin{pmatrix} \alpha_1 & 0 & \dots & \dots & 0 \\ 0 & \alpha_2 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \alpha_{n-1} & 0 \\ 0 & \dots & \dots & 0 & \alpha_n \end{pmatrix} \implies D^p = \begin{pmatrix} \alpha_1^p & 0 & \dots & \dots & 0 \\ 0 & \alpha_2^p & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \alpha_{n-1}^p & 0 \\ 0 & \dots & \dots & 0 & \alpha_n^p \end{pmatrix}$$

■

Theorem 1.1.4 A matrix A of size $n \times n$, triangular, is invertible if and only if its diagonal elements are not null.

Let A a matrix of size $n \times p$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1p} \\ a_{21} & a_{22} & \dots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{np} \end{pmatrix}.$$

Definition 1.1.7 We call **the transpose** of A the matrix A^T of size $p \times n$ defined by :

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1p} & a_{2p} & \dots & a_{np} \end{pmatrix}.$$

■ **Example 1.6**

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & -6 \\ -7 & 8 & 9 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 & -7 \\ 2 & 5 & 8 \\ 3 & -6 & 9 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 3 \\ 1 & -5 \\ -1 & 2 \end{pmatrix}^T = \begin{pmatrix} 0 & 1 & -1 \\ 3 & -5 & 2 \end{pmatrix} \qquad (1 \quad -2 \quad 5)^T = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$$

■

Theorem 1.1.5
1. $(A+B)^T = A^T + B^T$
2. $(\alpha A)^T = \alpha A^T$
3. $(A^T)^T = A$
4. $\boxed{(AB)^T = B^T A^T}$
5. If A is invertible, then A^T is also invertible, and we have $(A^T)^{-1} = (A^{-1})^T$.

(R) In the case of a square matrix of size $n \times n$, the elements a_{11}, $a_{22}, \dots, a_{nn}$ are called the **diagonal elements**.

Its **principal diagonal** is the diagonal $(a_{11}, a_{22}, \dots, a_{nn})$.

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

Definition 1.1.8 The **trace** of the matrix A is the sum of the diagonal elements of A. In other words,

$$\boxed{A = a_{11} + a_{22} + \cdots + a_{nn}.}$$

■ **Example 1.7**
- If $A = \begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix}$, then $(A) = 2 + 5 = 7$.
- For $B = \begin{pmatrix} 1 & 1 & 2 \\ 5 & 2 & 8 \\ 11 & 0 & -10 \end{pmatrix}$, $(B) = 1 + 2 - 10 = -7$.

■

Theorem 1.1.6 Let A and B two matrices $n \times n$. then :
1. $(A+B) = A + B$,
2. $(\alpha A) = \alpha\ A$ for all $\alpha \in \mathbb{K}$,
3. $(A^T) = A$,
4. $(AB) = (BA)$.

Definition 1.1.9 A matrix A of size $n \times n$ is **symmetric** if it is equal to its transpose, which means if

$$A = A^T,$$

or even, if $a_{ij} = a_{ji}$ for all $i, j = 1, \dots, n$. The coefficients are then symmetric and therefore symmetrical with respect to the diagonal.

■ **Example 1.8** The following matrices are symmetric :

$$\begin{pmatrix} 0 & 2 \\ 2 & 4 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 & 5 \\ 0 & 2 & -1 \\ 5 & -1 & 0 \end{pmatrix}$$

■

Determinants

1.2 Determinant

The determinant is a number we associate to n vectors $(v_1, \dots, v_n)$ of $\mathbb{R}^n$. It corresponds to the volume of parallelepiped generated by those n vectors. We can also define the determinant of a matrix A. The determinant lets you know if a matrix is invertible or not, and generally speaking, plays an important role in matrix calculus and linear systems resolution.

In all the following, we will consider matrices with coefficients in a commutative field $\mathbb{K}$, where the principal examples being $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We start by giving the expressions of the determinants for matrices in small dimensions.

In dimension 2, the determinant is easily calculated:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

It is the product of the elements of the principal diagonal minus the product of the second principal diagonal.
Let $A \in M_3(\mathbb{K})$ a matrix 3×3:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

This is the formula for the determinant:

$$\begin{aligned} \det A = \quad & +a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & -a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}\,. \end{aligned}$$

Let us calculate the determinant of the matrices :

$$A = \begin{pmatrix} 2 & 1 \\ -10 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

By calculus, we have:

$$\det A = 2 \times (-1) - (-10) \times 1 = 8.$$

$$\det B = 2 \times (-1) \times 1 + 1 \times 3 \times 3 + 0 \times 1 \times 2 \\ - 3 \times (-1) \times 0 - 2 \times 3 \times 2 - 1 \times 1 \times 1 = -6.$$

1.2.1 Calculus of determinants

One of the most useful technics to calculate the determinant is « Expansion by line (or by column)».

Definition 1.2.1 Let $A = (a_{ij}) \in M_n(\mathbb{K})$ a square matrix.

- We denote A_{ij} the extracted matrix, obtained by removing the line i and the column j of A.
- The number $\det A_{ij}$ is a **minor of order** $n-1$ of the matrix A.
- The number $C_{ij} = (-1)^{i+j} \det A_{ij}$ is the **cofactor** of A related to the coefficient a_{ij}.

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,j-1} & a_{1,j} & a_{1,j+1} & \cdots & a_{1,n} \\ \vdots & \vdots & & \vdots & & \vdots & \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i,1} & \cdots & a_{i,j-1} & a_{i,j} & a_{i,j+1} & \cdots & a_{i,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & \vdots & & \vdots & & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j} & a_{n,j+1} & \cdots & a_{nn} \end{pmatrix}$$

$$A_{ij} = \begin{pmatrix} a_{1,1} & \dots & a_{1,j-1} & a_{1,j+1} & \dots & a_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \dots & a_{i-1,j-1} & a_{i-1,j+1} & \dots & a_{i-1,n} \\ a_{i+1,1} & \dots & a_{i+1,j-1} & a_{i+1,j+1} & \dots & a_{i+1,n} \\ \vdots & & \vdots & & & \vdots \\ a_{n,1} & \dots & a_{n,j-1} & a_{n,j+1} & \dots & a_{n,n} \end{pmatrix}$$

■ **Example 1.9** Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Let's Calculate $A_{11}, C_{11}, A_{32}, C_{32}$.

$$A_{32} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 4 & 1 \end{pmatrix}.$$

$$C_{32} = (-1)^{3+2} \det A_{32} = (-1) \times (-11) = 11.$$

■

Remember that we associate signs : $A = \begin{pmatrix} + & - & + & - & \dots \\ - & + & - & + & \dots \\ + & - & + & - & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$.

So $C_{11} = +\det A_{11}$, $C_{12} = -\det A_{12}$, $C_{21} = -\det A_{21}$...

Theorem 1.2.1 — Expansion following a line or a column. Formula of expansion by line i:

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij}$$

Formula of expansion by column j:

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} = \sum_{i=1}^{n} a_{ij} C_{ij}$$

Let's find the formula of determinants 3×3, by expansion following the first line.

$$\begin{aligned}
\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\
&= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) \\
&\quad + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \\
&= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} + a_{12}a_{31}a_{23} - a_{12}a_{21}a_{33} \\
&\quad + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}.
\end{aligned}$$

Example 1.10

$$A = \begin{pmatrix} 4 & 0 & 3 & 1 \\ 4 & 2 & 1 & 0 \\ 0 & 3 & 1 & -1 \\ 1 & 0 & 2 & 3 \end{pmatrix}$$

$$\begin{aligned}
\det A &= 0C_{12} + 2C_{22} + 3C_{32} + 0C_{42} \\
&\qquad \text{(exp. through the second column)} \\
&= +2\begin{vmatrix} 4 & 3 & 1 \\ 0 & 1 & -1 \\ 1 & 2 & 3 \end{vmatrix} - 3\begin{vmatrix} 4 & 3 & 1 \\ 4 & 1 & 0 \\ 1 & 2 & 3 \end{vmatrix} \\
&\quad \text{exp. of two determinants } 3 \times 3 \\
&= +2\left(+4\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} - 0\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix}\right) \\
&\qquad \text{(following the first column)} \\
&\quad -3\left(-4\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} + 1\begin{vmatrix} 4 & 1 \\ 1 & 3 \end{vmatrix} - 0\begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix}\right) \\
&\qquad \text{(following the second column)} \\
&= +2\big(+4 \times 5 - 0 + 1 \times (-4)\big) \\
&\quad -3\big(-4 \times 7 + 1 \times 11 - 0\big) \\
&= 83
\end{aligned}$$

■

1.2.2 Calculus of the inverse of a matrix

Let $A \in M_n(\mathbb{K})$ a suqare matrix. We associate to it a matrix C of the cofactors, called **Comatrix**, denoted by $\mathrm{Com}(A)$:

$$C = (C_{ij}) = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}$$

Theorem 1.2.2 Let A a square matrix, then

$$A \text{ is invertible, if and only if} \det A \neq 0.$$

In addition, if A is invertible, and C is its comatrix, we can caluculate the inverse of A using the formula

$$\boxed{A^{-1} = \frac{1}{\det A} C^T}$$

■ **Example 1.11** Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

The calculus gives $\det A = 2$.
The comatrix C is obtained by calculating 9 determinants 2×2 (without forgetting the signs $+/-$). We find :

$$C = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix},$$

So

$$A^{-1} = \frac{1}{\det A} \cdot C^T = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}.$$

■

Matrix associated to a linear map (or Application)

1.3 Matrix associated to a linear map

We will see that there exists a close connection between matrices and linear applications. Naturally, to a matrix, we associate a linear application. Conversely, given a linear application and bases for the vector spaces of departure and arrival, we associate a matrix.

Let E and F be two $\mathbb{K}$-vector spaces of finite dimension. Let p be the dimension of E and $\mathscr{B} = (e_1, \ldots, e_p)$ be a basis for E. Let n be the dimension of F and $\mathscr{B}' = (f_1, \ldots, f_n)$ be a basis for F. Finally, let $f : E \to F$ be a linear application.

Definition 1.3.1 The **matrix of the linear application** f with respect to the bases $\mathscr{B}$ and $\mathscr{B}'$ is the matrix $(a_{i,j}) \in M_{n,p}(\mathbb{K})$ whose j-th column is formed by the coordinates of the vector $f(e_j)$ in the base $\mathscr{B}' = (f_1, f_2, \ldots, f_n)$:

$$\begin{array}{c} \\ f_1 \\ f_2 \\ \vdots \\ f_n \end{array} \begin{array}{c} \begin{array}{ccccc} f(e_1) & \ldots & f(e_j) & \ldots & f(e_p) \end{array} \\ \begin{pmatrix} a_{11} & & a_{1j} & \ldots & a_{1p} \\ a_{21} & & a_{2j} & \ldots & a_{2p} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & & a_{nj} & \ldots & a_{np} \end{pmatrix} \end{array}$$

In simpler terms: it is the matrix whose column vectors are the image by f of the vectors from the starting base $\mathscr{B}$, expressed in the destination base $\mathscr{B}'$. This matrix is denoted as $_{\mathscr{B},\mathscr{B}'}(f)$.

Let f be the linear application from $\mathbb{R}^3$ to $\mathbb{R}^2$ defined by

$$\begin{array}{rcl} f \quad : \quad \mathbb{R}^3 & \longrightarrow & \mathbb{R}^2 \\ (x_1, x_2, x_3) & \longmapsto & (x_1 + x_2 - x_3, x_1 - 2x_2 + 3x_3) \end{array}$$

It is useful to identify row vectors and column vectors; thus, f can be seen as the application $f : \left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} x_1+x_2-x_3 \\ x_1-2x_2+3x_3 \end{smallmatrix}\right)$.

Let $\mathscr{B} = (e_1, e_2, e_3)$ be the canonical basis of $\mathbb{R}^3$ and $\mathscr{B}' = (f_1, f_2)$ be the canonical basis of $\mathbb{R}^2$. That is:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad f_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad f_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

What is the matrix of f in the bases $\mathscr{B}$ and $\mathscr{B}'$?
We have $f(e_1) = f(1,0,0) = (1,1)$, $f(e_2) = (1,-2)$, and $f(e_3) = (-1,3)$. Thus:

$$_{\mathscr{B},\mathscr{B}'}(f) = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 3 \end{pmatrix}$$

We will now change the basis of the departure space and that of the arrival space. Consider the vectors

$$\varepsilon_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \varepsilon_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \varepsilon_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad \phi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \phi_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

One can easily show that $\mathscr{B}_0 = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ is a basis of $\mathbb{R}^3$ and $\mathscr{B}'_0 = (\phi_1, \phi_2)$ is a basis of $\mathbb{R}^2$.

What is the matrix of f in the bases $\mathscr{B}_0$ and $\mathscr{B}'_0$?

$f(\varepsilon_1) = f(1,1,0) = (2,-1) = 3\phi_1 - \phi_2$, $f(\varepsilon_2) = f(1,0,1) = (0,4) = -4\phi_1 + 4\phi_2$, $f(\varepsilon_3) = f(0,1,1) = (0,1) = -\phi_1 + \phi_2$, thus

$$_{\mathscr{B}_0,\mathscr{B}'_0}(f) = \begin{pmatrix} 3 & -4 & -1 \\ -1 & 4 & 1 \end{pmatrix}.$$

Conversely, given a matrix $A \in M_{2,3}(\mathbb{R})$ expressed in the canonical bases of $\mathbb{R}^2$ and $\mathbb{R}^3$ as

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

We can define the linear application $f \in \mathscr{L}(\mathbb{R}^3, \mathbb{R}^2)$ associated with it by

$$f(x,y,z) = (x+3y+5z, 2x+4y+6z)$$

Change of Basis, Transition Matrix

1.4 Change of Basis, Transition Matrix

Let E be a finite-dimensional vector space, and let $\mathscr{B} = (e_1, e_2, \dots, e_p)$ be a basis of E. For each $x \in E$, there exists a unique p-tuple of elements from $\mathbb{K}$ $(x_1, x_2, \dots, x_p)$ such that

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_p e_p.$$

The matrix of the coordinates of x is a column vector, denoted $_{\mathscr{B}}(x)$ or $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix}_{\mathscr{B}}$.

In $\mathbb{R}^p$, if $\mathscr{B}$ is the canonical basis, then we simply write $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix}$ without mentioning the basis.

Let E and F be two $\mathbb{K}$-vector spaces of finite dimension, and $f : E \to F$ a linear application.

Let $\mathscr{B}$ be a basis of E and $\mathscr{B}'$ a basis of F.

Proposition 1.4.1

- Let $A =_{\mathscr{B},\mathscr{B}'} (f)$.
- For $x \in E$, denote $X =_{\mathscr{B}} (x) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix}_{\mathscr{B}}$.
- For $y \in F$, denote $Y =_{\mathscr{B}'} (y) = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}_{\mathscr{B}'}$.

Then, if $y = f(x)$, we have

$$\boxed{Y = AX}$$

In other words:

$$\boxed{_{\mathscr{B}'}(f(x)) =_{\mathscr{B},\mathscr{B}'} (f) \times_{\mathscr{B}} (x)}$$

Functional Language	**Matrix Language**
Vector	Column Matrix
$x \in E$	$X \in M_{p,1}(\mathbb{K})$
0	$O_{p,1}$
$\lambda x + \mu y$	$\lambda X + \mu Y$
Linear Application	Rectangular Matrix
$u \in \mathscr{L}(E,F)$	$A \in M_{n,p}(\mathbb{K})$
$y = u(x)$	$Y = AX$
$v \circ u$	BA
u isomorphism, u^{-1}	A invertible, A^{-1}
$u \in \mathscr{L}(E)$	$A \in M_n(\mathbb{K})$
Id_E	I_n
u^m	A^m

Let E be a finite-dimensional vector space of dimension n. We know that all bases of E have n elements.

Definition 1.4.1 — Transition Matrix. Let $\mathscr{B}$ be a basis of E. Let $\mathscr{B}'$ be another basis of E.

The **Transition Matrix** from the base $\mathscr{B}$ to the base $\mathscr{B}'$, denoted ${}_{\mathscr{B},\mathscr{B}'}$, is the square matrix of size $n \times n$ whose j-th column is formed by the coordinates of the j-th vector of the base $\mathscr{B}'$, with respect to the base $\mathscr{B}$.

In summary:

The Transition Matrix ${}_{\mathscr{B},\mathscr{B}'}$ contains - in columns - the coordinates of the vectors of the new base $\mathscr{B}'$ expressed in the old base $\mathscr{B}$.

This is why sometimes ${}_{\mathscr{B},\mathscr{B}'}$ is also denoted as ${}_{\mathscr{B}}(\mathscr{B}')$.

Example 1.12 Let's consider the real vector space $\mathbb{R}^2$. We have

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \varepsilon_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \varepsilon_2 = \begin{pmatrix} 5 \\ 4 \end{pmatrix}.$$

Consider the base $\mathscr{B} = (e_1, e_2)$ and the base $\mathscr{B}' = (\varepsilon_1, \varepsilon_2)$.

What is the Transition Matrix from $\mathscr{B}$ to $\mathscr{B}'$?

We need to express ε_1 and ε_2 in terms of (e_1, e_2). We calculate that:

$$\varepsilon_1 = -e_1 + 2e_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}_{\mathscr{B}} \qquad \varepsilon_2 = e_1 + 4e_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}_{\mathscr{B}}$$

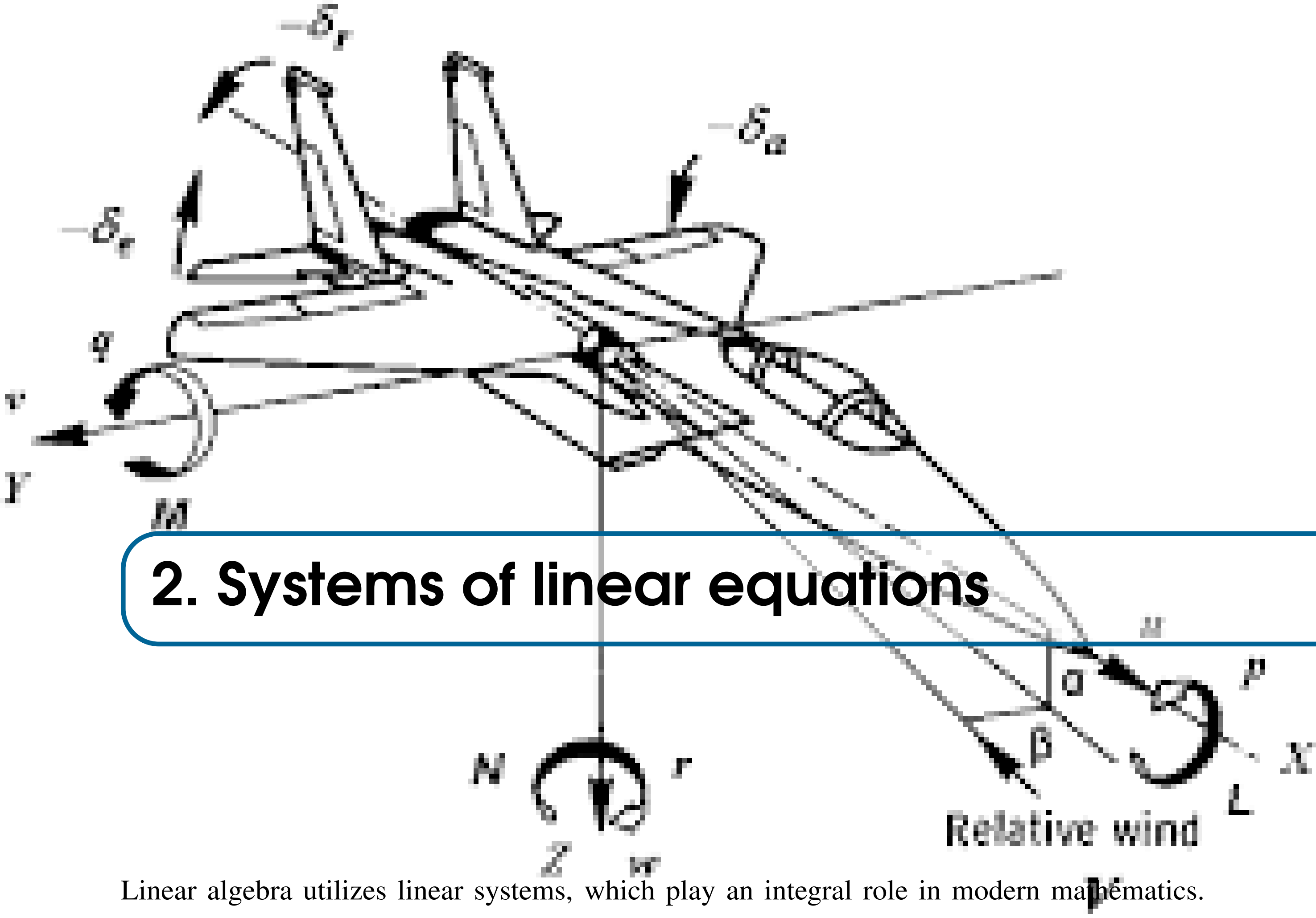
The Transition Matrix is therefore:

$$_{\mathscr{B},\mathscr{B}'} = \begin{pmatrix} -1 & 1 \\ 2 & 4 \end{pmatrix}$$

■

- Let E and F be two $\mathbb{K}$-vector spaces of finite dimension.
- Let $f : E \to F$ be a linear application.
- Let $\mathscr{B}_E$, $\mathscr{B}'_E$ be two bases of E.
- Let $\mathscr{B}_F$, $\mathscr{B}'_F$ be two bases of F.
- Let $P =_{\mathscr{B}_E,\mathscr{B}'_E}$ be the Transition Matrix from $\mathscr{B}_E$ to $\mathscr{B}'_E$.
- Let $Q =_{\mathscr{B}_F,\mathscr{B}'_F}$ be the Transition Matrix from $\mathscr{B}_F$ to $\mathscr{B}'_F$.
- Let $A =_{\mathscr{B}_E,\mathscr{B}_F} (f)$ be the matrix of the linear application f from the base $\mathscr{B}_E$ to the base $\mathscr{B}_F$.
- Let $B =_{\mathscr{B}'_E,\mathscr{B}'_F} (f)$ be the matrix of the linear application f from the base $\mathscr{B}'_E$ to the base $\mathscr{B}'_F$.

Theorem 1.4.2 — Change of Basis Formula. $\boxed{B = Q^{-1}AP}$



2. Systems of linear equations

Linear algebra utilizes linear systems, which play an integral role in modern mathematics. They are essential in various fields like engineering, physics, chemistry, computer science, and economics. A linear system is frequently used as an approximation for a non-linear system of equations.

INTRODUCTION TO SYSTEMS OF LINEAR EQUATIONS

2.1 Introduction to Systems of linear equations

Let $n,\ p\ \in \mathbb{N}^*$. We call **Linear Systems** of n equations with p unknowns, any equations system of the form

$$(\mathscr{S})\left\{\begin{array}{cccccccc} a_{11}x_1 & +a_{12}x_2 & +a_{13}x_3 & + & \cdots & +a_{1p}x_p & = & b_1 & (\leftarrow \text{equation } 1) \\ a_{21}x_1 & +a_{22}x_2 & +a_{23}x_3 & + & \cdots & +a_{2p}x_p & = & b_2 & (\leftarrow \text{equation } 2) \\ \vdots & \vdots & \vdots & & & \vdots & = & \vdots & \\ a_{i1}x_1 & +a_{i2}x_2 & +a_{i3}x_3 & + & \cdots & +a_{ip}x_p & = & b_i & (\leftarrow \text{equation } i) \\ \vdots & \vdots & \vdots & & & \vdots & = & \vdots & \\ a_{n1}x_1 & +a_{n2}x_2 & +a_{n3}x_3 & + & \cdots & +a_{np}x_p & = & b_n & (\leftarrow \text{equation } n) \end{array}\right.$$

where, for $i = 1,\dots,n,\ j = 1,\dots,p$, the scalars $a_{ij}, b_i \in \mathbb{K}$ are given.

For all $i \in 1,n, j \in 1,p$, the x_j are called the **unknowns of the system**, whereas the a_{ij} are called **coefficients of the system** and the b_j form the **second member of the system**.

- $\begin{cases} x_1 - 2x_2 = 1 \\ 2x_1 - 3x_2 = 4 \end{cases}$ is a linear system of 2 equations with 2 unknowns. The second member is $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$.
- $\begin{cases} x + 2y + z = 3 \\ 7x - 5y - 2z = 2 \end{cases}$ is a linear system of 2 equations with 3 unknowns. The second member is $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.
- $\begin{cases} e^x + y = 1 \\ x + \sin(y) = 2 \end{cases}$ is not a linear system.
- $\begin{cases} x^2 + 2y^3 = -3 \\ 2x^4 - y^5 = 2 \end{cases}$ is not a linear system.
- Let $(\mathscr{S})$ the linear system introduced previously.
- We call **matrix of a system** $(\mathscr{S})$ the matrix A of n lines and p columns given by

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,j} & \dots & a_{1,p} \\ a_{2,1} & a_{2,2} & \dots & a_{2,j} & \dots & a_{2,p} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i,1} & a_{i,2} & \dots & a_{i,j} & \dots & a_{i,p} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,j} & \dots & a_{n,p} \end{pmatrix} \in \mathbb{M}_{n,p}(\mathbb{K}).$$

- We call **matrix column of the second member** of $(\mathscr{S})$ the vector column B constituted by the coefficients $b_j,\ j = 1, \dots, p$ defined by :

$$B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{M}_{n,1}(\mathbb{K}).$$

- We call **augmented system matrix** $(\mathscr{S})$ the matrix $\tilde{A} = (A|B)$ of n lines and $p+1$ columns given by

$$\tilde{A} = \left(\begin{array}{cccccc|c} a_{1,1} & a_{1,2} & \dots & a_{1,j} & \dots & a_{1,p} & b_1 \\ a_{2,1} & a_{2,2} & \dots & a_{2,j} & \dots & a_{2,p} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i,1} & a_{i,2} & \dots & a_{i,j} & \dots & a_{i,p} & b_i \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,j} & \dots & a_{n,p} & b_n \end{array}\right) \in \mathbb{M}_{n,p+1}(\mathbb{K}).$$

- If we note $X = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} \in \mathbb{M}_{p,1}(\mathbb{K})$ then the system $(\mathscr{S})$ is equivalent to

$$A.X = B.$$

This equation is called system matrix equation, and the rank of A is also called rank of the system.

Given the following system

■ **Example 2.1** Given the following system
$(\mathscr{S}) : \begin{cases} x = y + 2z + 1 \\ z + y + 2 = -3x \\ 2x + 3y - 3 = 17z \end{cases}$ then the associated matrix to $(\mathscr{S})$ is $A = \begin{pmatrix} 1 & -1 & -2 \\ 3 & 1 & 1 \\ 2 & 3 & -17 \end{pmatrix}$

The matrix column of the second member is $B = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and the augmented matrix is

$\tilde{A} = \left(\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 3 & 1 & 1 & -2 \\ 2 & 3 & -17 & 3 \end{array}\right).$

Hence $(\mathscr{S})$ is written

$$\begin{pmatrix} 1 & -1 & -2 \\ 3 & 1 & 1 \\ 2 & 3 & -17 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}.$$

■

STUDY OF THE SET OF SOLUTIONS

2.2 Study of the set of solutions

Definition 2.2.1 A **solution** of a linear system is a p-tuple $(s_1, s_2, \ldots, s_p) \in \mathbb{K}^p$ (p numbers (reals or complexes) such that if we substitute x_1 by s_1, x_2 by s_2, ... and x_p by s_p in a linear system described in the definition, we get an equality. The **set of solutions of the system** is the set of all its p-tuples.

Example 2.2 The system

$$\begin{cases} x_1 - 3x_2 + x_3 = 1 \\ -2x_1 + 4x_2 - 3x_3 = 9 \end{cases}$$

admits a solution $(-18, -6, 1)$, in other words

$$x_1 = -18\,, \qquad x_2 = -6\,, \qquad x_3 = 1\,.$$

However, $(7, 2, 0)$ is not a solution of the system. ■

We aim to determine all the solutions of a linear system. This is called **solving** the system.

Theorem 2.2.1 For a system of linear equations, three cases arise:

1. The system does not admit any solution.
2. The system admits only one solution.
3. The system admits an infinity of solutions.

R A linear system which admits at least one solution is said to be **compatible** otherwise, it is said to be **incompatible**.
An important special case is that of **homogeneous systems** (systems whose second member is zero). Such systems are always compatible because they admit the solution $s_1 = s_2 = \cdots = s_p = 0$. This solution is called *trivial solution*.

METHODS FOR SOLVING A LINEAR SYSTEM

2.3 Methods for solving a linear system

2.3.1 Operations on the equations of a linear system

We will define three elementary operations on the equations (that is to say on the lines) which are:

Definition 2.3.1 — Elementary operations. We denote by $\mathrm{L}_1, \ldots, \mathrm{L}_n$ the lines of a linear system of n equations and let $(i, j) \in 1, n^2$ and $\lambda \in \mathbb{K}^*$.

- We denote $L_i \leftrightarrow L_j$, the operation consisting of exchanging (permuting) the lines L_i and L_j.
- We note $\mathrm{L}_i \leftarrow \lambda \mathrm{L}_i$, the operation consisting of multiplying the line L_i by λ.
- We note $\mathrm{L}_i \leftarrow \mathrm{L}_i + \lambda \mathrm{L}_j$, the operation consisting of adding λ times line L_j to line L_i.

Note: each of these 3 operations results in a system equivalent to the first i.e. a system having the same compatibility and the same set of solutions.

Let $(\mathscr{S}):\left\{\begin{array}{rrrcrl} x & +y & +7z & = & -1 & (L_1) \\ 2x & -y & +5z & = & -5 & (L_2) \\ -x & -3y & -9z & = & -5 & (L_3) \end{array}\right.$

The result of $L_1 \leftarrow -3L_1$ is $(\mathscr{S}'):\left\{\begin{array}{rrrcr} -3x & -3y & -21z & = & 3 \\ 2x & -y & +5z & = & -5 \\ -x & -3y & -9z & = & -5 \end{array}\right.$. If we apply the operation $L_2 \leftrightarrow L_3$ to $(\mathscr{S}')$, we get $(\mathscr{S}'')$:

$$(\mathscr{S}'')\left\{\begin{array}{rrrcr} -3x & -3y & -21z & = & 3 \\ -x & -3y & -9z & = & -5 \\ 2x & -y & +5z & = & -5 \end{array}\right. .$$

By the operation $L_3 \leftarrow L_3 - 2L_2$ applied to $(\mathscr{S}'')$, we get $(\mathscr{S}''')$:

$$(\mathscr{S}''')\left\{\begin{array}{rrrcr} -3x & -3y & -21z & = & 3 \\ -x & -3y & -9z & = & -5 \\ 4x & +5y & +23z & = & 5 \end{array}\right.$$

2.3.2 Resolution by substitution

To find out if there are one or more solutions to a linear system, and to calculate them, a first method is **substitution**. For example for the system:

$$\left\{\begin{array}{rcl} 3x+2y & = & 1 \\ 2x-7y & = & -2 \end{array}\right. \qquad (S)$$

We rewrite the first line $3x+2y=1$ as $y=\frac{1}{2}-\frac{3}{2}x$, and we replace (we *substitute*) the y of the second equation, by the expression $\frac{1}{2}-\frac{3}{2}x$. We get an equivalent system :

$$\left\{\begin{array}{rcl} y & = & \frac{1}{2}-\frac{3}{2}x \\ 2x-7(\frac{1}{2}-\frac{3}{2}x) & = & -2 \end{array}\right.$$

The second equation is now an expression that contains only x, and we can solve it:

$$\left\{\begin{array}{rcl} y & = & \frac{1}{2}-\frac{3}{2}x \\ (2+7\times\frac{3}{2})x & = & -2+\frac{7}{2} \end{array}\right. \iff \left\{\begin{array}{rcl} y & = & \frac{1}{2}-\frac{3}{2}x \\ x & = & \frac{3}{25} \end{array}\right.$$

Remember :

$$\begin{cases} y &= \frac{1}{2} - \frac{3}{2}x \\ (2+7\times\frac{3}{2})x &= -2+\frac{7}{2} \end{cases} \iff \begin{cases} y &= \frac{1}{2} - \frac{3}{2}x \\ x &= \frac{3}{25} \end{cases}$$

All that remains is to replace the value of x obtained in the first line:

$$\begin{cases} y &= \frac{8}{25} \\ x &= \frac{3}{25} \end{cases}$$

The system (S) admits a unique solution $(\frac{3}{25}, \frac{8}{25})$. The set of solutions is therefore

$$\mathscr{S} = \left\{ \left(\frac{3}{25}, \frac{8}{25} \right) \right\}.$$

2.3.3 Inverse matrix method

In matrix terms, the linear system

$$\begin{cases} ax+by &= e \\ cx+dy &= f \end{cases}$$

is equivalent to

$$AX = Y \quad \text{où} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad Y = \begin{pmatrix} e \\ f \end{pmatrix}.$$

If the determinant of the matrix A is not null, which means if $ad-bc \neq 0$, then the matrix A is invertible and

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

and the unique solution $X = \binom{x}{y}$ of the system is given by

$$X = A^{-1}Y.$$

■ **Example 2.3** Let's solve the system $\begin{cases} x+y &= 1 \\ x+t^2y &= t \end{cases}$ following the argument $t \in \mathbb{R}$.

the determinant of the system is $\begin{vmatrix} 1 & 1 \\ 1 & t^2 \end{vmatrix} = t^2 - 1$.

First case: $t \neq +1$ and $t \neq -1$. Then $t^2 - 1 \neq 0$. The matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & t^2 \end{pmatrix}$ is invertible with for inverse $A^{-1}=\frac{1}{t^2-1}\left(\begin{smallmatrix} t^2 & -1 \\ -1 & 1 \end{smallmatrix}\right)$. we have:

$$X=\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)=A^{-1}B=\frac{1}{t^2-1}\begin{pmatrix} t^2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ t \end{pmatrix}=\frac{1}{t^2-1}\begin{pmatrix} t^2-t \\ t-1 \end{pmatrix}=\begin{pmatrix} \frac{t}{t+1} \\ \frac{1}{t+1} \end{pmatrix}.$$

For each $t\neq\pm1$, the set of solutions is $\mathscr{S}=\left\{\left(\frac{t}{t+1},\frac{1}{t+1}\right)\right\}$.

Second case: $t=+1$. There are an infinity of solutions : $\mathscr{S}=\{(x,1-x)\mid x\in\mathbb{R}\}$ (identical equations).

Third case: $t=-1$. The system is then written : $\begin{cases} x+y &= 1 \\ x+y &= -1 \end{cases}$, The two equations are clearly incompatible and then $\mathscr{S}=\varnothing$. ■

Consider the following system of linear equations with n equations and n unknowns:

$$\begin{cases} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n &= b_1 \\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n &= b_2 \\ \qquad\qquad\cdots \\ a_{n1}x_1+a_{n2}x_2+\cdots+a_{nn}x_n &= b_n \end{cases}$$

or in matrix form $AX=B$ with

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\in M_n(\mathbb{K}), X=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ et } B=\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Theorem 2.3.1 Under the condition $\det(A)\neq0$, the solution of the previous system is given by the formula:

$$X=A^{-1}B.$$

2.3.4 Cramer's method

We consider the case of a system of 2 equations with 2 unknowns:

$$\begin{cases} ax+by &= e \\ cx+dy &= f \end{cases}$$

We denote $\Delta = \left|\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right| = ad - bc$ the **determinant**.
If $\Delta \neq 0$, we find a unique solution (x,y) such that:

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\Delta} \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\Delta}$$

Note that the denominator equals the determinant for both coordinates and is therefore non-zero. For the numerator of the first coordinate x, we replace the first column by the second member; for the second coordinate y, we replace the second column by the second member.

■ **Example 2.4** Let's solve the system $\begin{cases} tx - 2y &= 1 \\ 3x + ty &= 1 \end{cases}$ depending on the parameter value $t \in \mathbb{R}$.

The determinant associated to the system is $\left|\begin{smallmatrix} t & -2 \\ 3 & t \end{smallmatrix}\right| = t^2 + 6$ and never cancels. Therefore, there exists a unique solution (x,y) and it satisfies:

$$x = \frac{\begin{vmatrix} 1 & -2 \\ 1 & t \end{vmatrix}}{t^2+6} = \frac{t+2}{t^2+6}, \qquad y = \frac{\begin{vmatrix} t & 1 \\ 3 & 1 \end{vmatrix}}{t^2+6} = \frac{t-3}{t^2+6}.$$

For each t, the set of solutions is $\mathscr{S} = \left\{ \left(\frac{t+2}{t^2+6}, \frac{t-3}{t^2+6} \right) \right\}$. ■

Definition 2.3.2 — Cramer's System. A system of linear equations with as many equations as unknowns and whose determinant of the coefficient matrix is non-zero is called a Cramer's system.

Let us define the matrix $A_j \in M_n(\mathbb{K})$ by

$$A_j = \begin{pmatrix} a_{11} & \dots & a_{1,j-1} & b_1 & a_{1,j+1} & \dots & a_{1n} \\ a_{21} & \dots & a_{2,j-1} & b_2 & a_{2,j+1} & \dots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \dots & a_{n,j-1} & b_n & a_{n,j+1} & \dots & a_{nn} \end{pmatrix}$$

In other words, A_j is the matrix obtained by replacing the j-th column of A by the second member B.

Theorem 2.3.2 — Cramer's rule. Let $AX = B$ a Cramer's system, i.e.
a system of n equations with n unknowns with A invertible.
So this system has one and only one solution $(x_1, x_2, \dots, x_n)$ given by:

$$x_1 = \frac{\det A_1}{\det A} \qquad x_2 = \frac{\det A_2}{\det A} \qquad \dots \qquad x_n = \frac{\det A_n}{\det A}.$$

■ **Example 2.5** Let's solve the system:

$$\begin{cases} x_1 & & + & 2x_3 & = & 6 \\ -3x_1 & + & 4x_2 & + & 6x_3 & = & 30 \\ -x_1 & - & 2x_2 & + & 3x_3 & = & 8. \end{cases} \quad A = \begin{pmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{pmatrix} B = \begin{pmatrix} 6 \\ 30 \\ 8 \end{pmatrix}.$$

We have

$$A_1 = \begin{pmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{pmatrix} A_2 = \begin{pmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{pmatrix} A_3 = \begin{pmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{pmatrix}$$

and

$$\det A = 44 \qquad \det A_1 = -40 \qquad \det A_2 = 72 \qquad \det A_3 = 152.$$

The solution is then

$$x_1 = \frac{\det A_1}{\det A} = -\frac{40}{44} \quad x_2 = \frac{\det A_2}{\det A} = \frac{72}{44} \quad x_3 = \frac{\det A_3}{\det A} = \frac{152}{44}.$$

■

Gaussian method

2.3.5 Gaussian method

The Gaussian pivot method allows you to find solutions to any linear system. We will describe this algorithm using an example. It is a precise description of a sequence of operations to be carried out, which depend on the situation and a precise order. This process always results (and rather quickly) in a scaled and then reduced system, which immediately leads to the system solutions.

Consider the following system to solve:

$$\begin{cases} & -x_2 & +2x_3 & +13x_4 & = & 5 \\ x_1 & -2x_2 & +3x_3 & +17x_4 & = & 4 \\ -x_1 & +3x_2 & -3x_3 & -20x_4 & = & -1 \end{cases}$$

To apply the Gaussian pivot method, it is necessary that the first coefficient of the first line is non-zero. As this is not the case here, we exchange the first two lines using the elementary operation $L_1 \leftrightarrow L_2$:

$$\begin{cases} x_1 & -2x_2 & +3x_3 & +17x_4 & = & 4 & \quad L_1 \leftrightarrow L_2 \\ & -x_2 & +2x_3 & +13x_4 & = & 5 \\ -x_1 & +3x_2 & -3x_3 & -20x_4 & = & -1 \end{cases}$$

We already have a coefficient 1 in front of the x_1 in the first line. We say that we have a **pivot** in position $(1,1)$ (first row, first column). This pivot serves as the basis for eliminating all other terms on the same column.

There is no x_1 term on the second line. Let's remove the term x_1 from the third line; for this we perform the elementary operation $L_3 \leftarrow L_3 + L_1$:

$$\left\{\begin{array}{cccccc} x_1 & -2x_2 & +3x_3 & +17x_4 & = & 4 \\ & -x_2 & +2x_3 & +13x_4 & = & 5 \\ & x_2 & & -3x_4 & = & 3 \end{array}\right. \quad {\scriptstyle L_3 \leftarrow L_3 + L_1}$$

We change the sign of the second line ($L_2 \leftarrow -L_2$) to show 1 to the pivot coefficient $(2,2)$ (second row, second column):

$$\left\{\begin{array}{cccccc} x_1 & -2x_2 & +3x_3 & +17x_4 & = & 4 \\ & x_2 & -2x_3 & -13x_4 & = & -5 \\ & x_2 & & -3x_4 & = & 3 \end{array}\right. \quad {\scriptstyle L_2 \leftarrow -L_2}$$

We remove the term x_2 from the third line, then we reveal a coefficient 1 for the pivot of the position $(3,3)$:

$$\left\{\begin{array}{cccccc} x_1 & -2x_2 & +3x_3 & +17x_4 & = & 4 \\ & x_2 & -2x_3 & -13x_4 & = & -5 \\ & & 2x_3 & +10x_4 & = & 8 \end{array}\right. \quad {\scriptstyle L_3 \leftarrow L_3 - L_2}$$

$$\left\{\begin{array}{cccccc} x_1 & -2x_2 & +3x_3 & +17x_4 & = & 4 \\ & x_2 & -2x_3 & -13x_4 & = & -5 \\ & & x_3 & +5x_4 & = & 4 \end{array}\right. \quad {\scriptstyle L_3 \leftarrow \frac{1}{2} L_3}$$

The system is now in phased (staggered) form.

It remains to put it in reduced scale form. To do this, we add to a line suitable multiples of the lines located below it, going from bottom right to top left. We make 0 appear on the third column using the pivot of the third row:

$$\left\{\begin{array}{cccccc} x_1 & -2x_2 & +3x_3 & +17x_4 & = & 4 \\ & x_2 & & -3x_4 & = & 3 \\ & & x_3 & +5x_4 & = & 4 \end{array}\right. \quad {\scriptstyle L_2 \leftarrow L_2 + 2L_3}$$

and

$$\left\{\begin{array}{cccccc} x_1 & -2x_2 & & 2x_4 & = & -8 \\ & x_2 & & -3x_4 & = & 3 \\ & & x_3 & +5x_4 & = & 4 \end{array}\right. \quad {\scriptstyle L_1 \leftarrow L_1 - 3L_3}$$

We make 0 appear on the second column (using the pivot of the second row):

$$\begin{cases} x_1 & & -4x_4 & = & -2 & \quad L_1 \leftarrow L_1 + 2L_2 \\ & x_2 & -3x_4 & = & 3 & \\ & & x_3 + 5x_4 & = & 4 & \end{cases}$$

The system is now in a reduced form.

The system is now very simple to solve. By choosing x_4 as a free variable, we can express x_1, x_2, x_3 in terms of x_4 :

$$x_1 = 4x_4 - 2, \quad x_2 = 3x_4 + 3, \quad x_3 = -5x_4 + 4.$$

Which makes it possible to obtain all the solutions of the system:

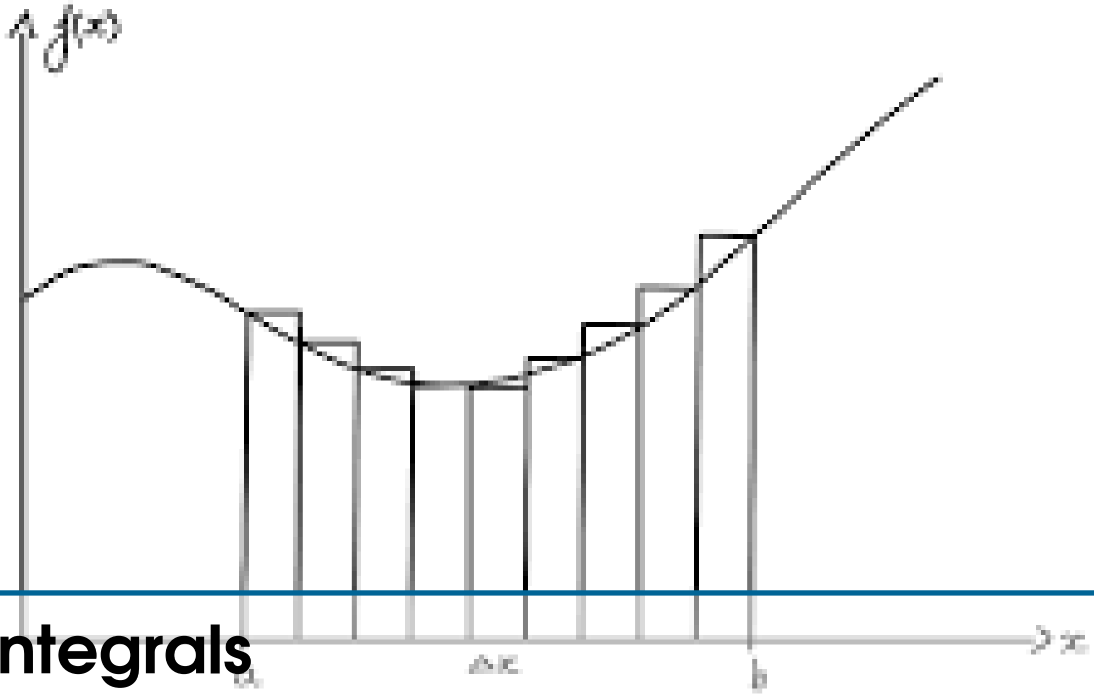
$$\mathscr{S} = \big\{(4x_4 - 2, 3x_4 + 3, -5x_4 + 4, x_4) \mid x_4 \in \mathbb{R}\big\}.$$

The Gaussian pivot method works as follows:

- We are looking for a line showing the first unknown. The coefficient appearing in front of this unknown is called the pivot. We swap lines to bring the pivot to the first line.
- We make the first unknown disappear from the other lines using elementary operations $L_i \leftarrow L_i + \lambda L_1$.
- We start again from the second line and the next unknown which still appears in the following lines.

We then arrive at a staggered system of the following form:

$$\begin{cases} a'_{1,j_1} x_{j_1} & + & \cdots & + & \cdots & + & \cdots & + & \cdots & + & \cdots & + & \cdots & = & b'_1 \\ & & & & a'_{2,j_2} x_{j_2} & + & \cdots & + & \cdots & + & \cdots & + & \cdots & = & b'_2 \\ & & & & & \ddots & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & & & & & a'_{r,j_r} x_{j_r} & + & \cdots & + & \cdots & = & b'_r \\ & & & & & & & & & & & & 0 & = & b'_{r+1} \\ & & & & & & & & & & & & \vdots & \vdots & \vdots \\ & & & & & & & & & & & & 0 & = & b'_n \end{cases}$$



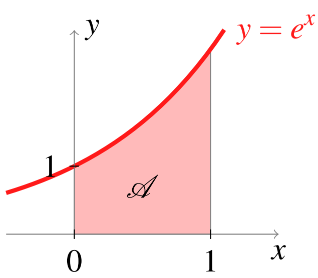
3. Integrals

In mathematics, an integral is the continuous analog of a sum, which is used to calculate areas, volumes, and their generalizations. Integration, the process of computing an integral, is one of the two fundamental operations of calculus, the other being differentiation.

DEFINITE INTEGRAL

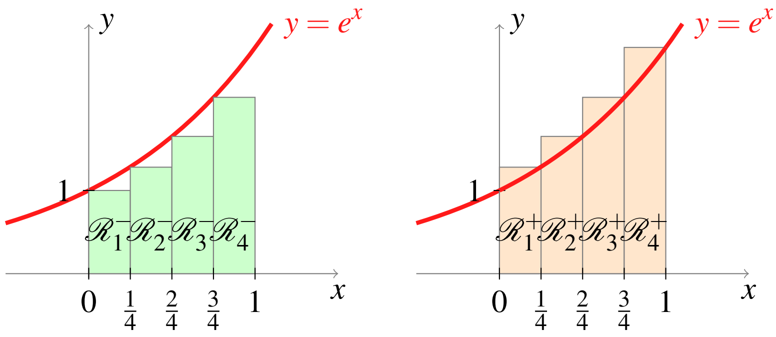
3.1 Definite integral

We will introduce the integral using an example. Consider the exponential function $f(x) = e^x$. We want to calculate the area $\mathscr{A}$ below the graph of f and between the equation lines $(x = 0)$, $(x = 1)$ and the axis (Ox).



We approximate this area by sums of areas of the rectangles located under the curve. More precisely, let $n \geq 1$ an integer; let's cut our interval $[0, 1]$ using subdivision $(0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{i}{n}, \cdots, \frac{n-1}{n}, 1)$.

We consider the «lower rectangles » $\mathscr{R}_i^-$, each based on the interval $[\frac{i-1}{n}, \frac{i}{n}]$ and for height $f(\frac{i-1}{n}) = e^{(i-1)/n}$. The integer i varies from 1 to n. The area of $\mathscr{R}_i^-$ is «base $\times$ height » : $(\frac{i}{n} - \frac{i-1}{n}) \times e^{(i-1)/n} = \frac{1}{n} e^{\frac{i-1}{n}}$.



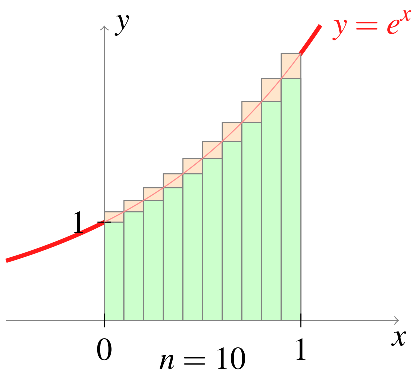
The sum of the areas of $\mathscr{R}_i^-$ is then calculated as the sum of a geometric sequence:

$$\sum_{i=1}^{n} \frac{e^{\frac{i-1}{n}}}{n} = \frac{1}{n} \sum_{i=1}^{n} \left(e^{\frac{1}{n}}\right)^{i-1} = \frac{1}{n} \frac{1 - \left(e^{\frac{1}{n}}\right)^n}{1 - e^{\frac{1}{n}}} = \frac{\frac{1}{n}}{e^{\frac{1}{n}} - 1} (e-1) \xrightarrow[n \to +\infty]{} e - 1.$$

For the limit we recognized the expression of the type $\frac{e^x - 1}{x} \xrightarrow[x \to 0]{} 1$ (avec ici $x = \frac{1}{n}$).

Let now be the «upper rectangles » $\mathscr{R}_i^+$, having the same base $[\frac{i-1}{n}, \frac{i}{n}]$ but the height $f(\frac{i}{n}) = e^{i/n}$. A similar calculation shows that $\sum_{i=1}^{n} \frac{e^{\frac{i}{n}}}{n} \to e - 1$ when $n \to +\infty$.

The area $\mathscr{A}$ of our region is greater than the sum of the areas of the lower rectangles; and it is lower to the sum of the areas of the upper rectangles.



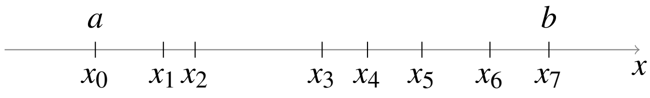
When considering smaller and smaller subdivisions (that is to say when we make n tend towards $+\infty$) then we obtain in the limit that the area $\mathscr{A}$ of our region is framed by two areas which tend towards $e - 1$. So the area of our region is $\mathscr{A} = e - 1$.

We will repeat this construction for any function f.

What will replace the rectangles will be ***step functions***.

If the limit of the areas below equals the limit of the areas above we call this common limit ***the definite integral*** of f which we note $\int_a^b f(x)\,dx$.
However it is not always true that these limits are equal, the integral is therefore only defined for ***integrable*** functions. Fortunately we will see that if the function f is continuous then it is integrable.

Definition 3.1.1 Let $[a,b]$ a closed interval bounded in $\mathbb{R}$ $(-\infty < a < b < +\infty)$. We call a **subdivision** of $[a,b]$ a finite sequence, strictly increasing, of numbers $\mathscr{S} = (x_0, x_1, \ldots, x_n)$ such that $x_0 = a$ and $x_n = b$. In other words $a = x_0 < x_1 < \cdots < x_n = b$.

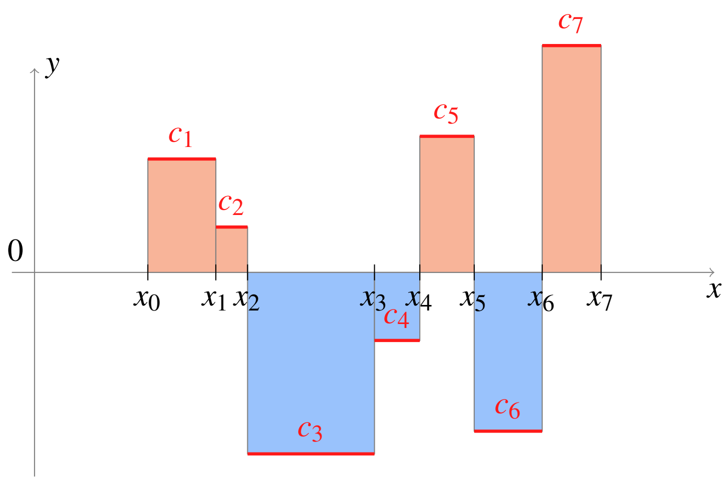


Definition 3.1.2 A function $f : [a,b] \to \mathbb{R}$ is a **step function** if there is a subdivision $(x_0, x_1, \ldots, x_n)$ and real numbers $c_1, \ldots, c_n$ such as for every $i \in \{1, \ldots, n\}$ we get

$$\forall x \in]x_{i-1}, x_i[\quad f(x) = c_i$$

In other words f is a constant function on each of the subintervals of the subdivision.

R The value of f at points x_i of the subdivision is not imposed. It can be equal to that of the interval which precedes or that which follows, or another arbitrary value. This doesn't matter because the area won't change.



Definition 3.1.3 For a step function like above, its **integral** is the real $\int_a^b f(x)\,dx$ defined by

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} c_i (x_i - x_{i-1})$$

R Note that each term $c_i(x_i - x_{i-1})$ is the area of the rectangle between the abscissa x_{i-1} and x_i and height c_i. You just have to be careful that you count the area with a sign «+ » if $c_i > 0$ and a sign «− » if $c_i < 0$.

The integral of a step function is the area of the part located above of the abscissa axis (here in red) minus the area of the part located below (in blue). The integral of a step function is indeed a real number which measures the area between the curve of f and the x-axis.

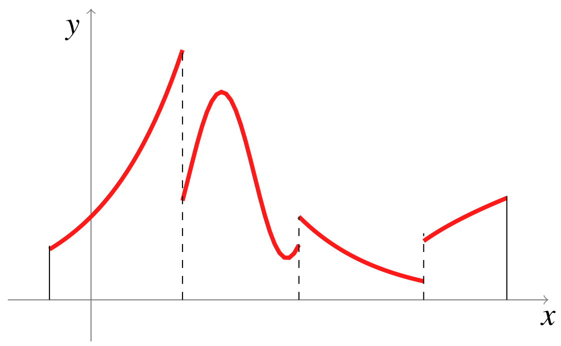
This is the most important theoretical result of this chapter.

Theorem 3.1.1 If $f : [a,b] \to \mathbb{R}$ is continuous then f is integrable.

The idea is that continuous functions can be approached as closely as we want by step functions, while keeping uniform error control over the interval.

A function $f : [a,b] \to \mathbb{R}$ is said to be **piecewise continuous** if there exists an integer n and a subdivision $(x_0,\dots,x_n)$ such that $f_{|]x_{i-1},x_i[}$ is continuous, admits a finite limit on the right in x_{i-1} and a left limit in x_i for all $i \in \{1,\dots,n\}$.

R Piecewise continuous functions are integrable.



Here is a result which proves that we can also integrate functions which are not continuous provided that the function is increasing (or decreasing).

Theorem 3.1.2 If $f : [a,b] \to \mathbb{R}$ is monotone then f is integrable.

PRIMITIVE / INTEGRAL

3.2 Indefinite integral

Definition 3.2.1 Let $f : I \to \mathbb{R}$ be a function defined on any interval I. We say that $F : I \to \mathbb{R}$ is a **primitive** of f on I if F is a differentiable function on I checking $F'(x) = f(x)$ for all $x \in I$.
The set of primitives of f is called **indefinite integral** of f.

■ **Example 3.1** 1. Let $I = \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Then $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = \dfrac{x^3}{3}$ is a primitive of f. The function defined by $F(x) = \dfrac{x^3}{3} + 1$ is also a primitive of f.

2. Let $I = [0, +\infty[$ and $g : I \to \mathbb{R}$ defined by $g(x) = \sqrt{x}$. Then $G : I \to \mathbb{R}$ defined by $G(x) = \frac{2}{3}x^{\frac{3}{2}}$ is a primitive of g on I. For all $c \in \mathbb{R}$, the function $G + c$ is also a primitive of g.

■

Proposition 3.2.1 Let $f : I \to \mathbb{R}$ a function and either $F : I \to \mathbb{R}$ a primitive of f. Every primitive of f is written $G = F + c$ where $c \in \mathbb{R}$.

Notations. We will note a primitive of f by $\int f(t)\,dt$ or $\int f(x)\,dx$ or $\int f(u)\,du$ (the letters $t, x, u, \ldots$ are so-called *silent* letters, i.e. interchangeable).

The above proposition tells us that if F is a primitive of f then there exists a real c, such that $F = \int f(t)\,dt + c$.

By derivation we easily prove the following result:

Proposition 3.2.2 Let F be a primitive of f and G be a primitive of g. Then $F + G$ is a primitive of $f + g$. And if $\lambda \in \mathbb{R}$ then λF is a primitive of λf.

$\int e^x\,dx = e^x + c$ on $\mathbb{R}$
$\int \cos x\,dx = \sin x + c$ on $\mathbb{R}$
$\int \sin x\,dx = -\cos x + c$ on $\mathbb{R}$
$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad (n \in \mathbb{N})$ on $\mathbb{R}$
$\int x^\alpha\,dx = \frac{x^{\alpha+1}}{\alpha+1} + c \quad (\alpha \in \mathbb{R} \setminus \{-1\})$ on $]0, +\infty[$
$\int \frac{1}{x}\,dx = \ln\|x\| + c$ on $]0, +\infty[$ or $]-\infty, 0[$

$\int \sinh x\,dx = \cosh x + c$, $\int \cosh x\,dx = \sinh x + c$	on $\mathbb{R}$
$\int \frac{dx}{1+x^2} = \arctan x + c$	on $\mathbb{R}$
$\int \frac{dx}{\sqrt{1-x^2}} = \begin{cases} \arcsin x + c \\ \frac{\pi}{2} - \arccos x + c \end{cases}$	on $]-1, 1[$
$\int \frac{dx}{\sqrt{x^2+1}} = \begin{cases} \operatorname{Argsh} x + c \\ \ln\left(x + \sqrt{x^2+1}\right) + c \end{cases}$	on $\mathbb{R}$
$\int \frac{dx}{\sqrt{x^2-1}} = \begin{cases} \operatorname{Argch} x + c \\ \ln\left(x + \sqrt{x^2-1}\right) + c \end{cases}$	on $x \in]1, +\infty[$

R

These primitives should be known by heart.

1. Here's how to read this table. If f is the function defined on $\mathbb{R}$ by $f(x) = x^n$ then the function: $x \mapsto \dfrac{x^{n+1}}{n+1}$ is a primitive of f on $\mathbb{R}$. The primitives of f are the functions defined by $x \mapsto \dfrac{x^{n+1}}{n+1} + c$ (for c any real constant). And we write $\int x^n\, dx = \dfrac{x^{n+1}}{n+1} + c$, where $c \in \mathbb{R}$.
2. Remember that the variable under the integral symbol is a silent variable. We can also write $\int t^n\, dt = \dfrac{x^{n+1}}{n+1} + c$.
3. We can find primitives with very different appearances, for example $x \mapsto \arcsin x$ and $x \mapsto \dfrac{\pi}{2} - \arccos x$ are two primitives of the same function $x \mapsto \dfrac{1}{\sqrt{1-x^2}}$. But of course we know that $\arcsin x + \arccos x = \pi/2$.

Theorem 3.2.3 Let $f : [a,b] \to \mathbb{R}$ be a continuous function. The function $F : I \to \mathbb{R}$ defined by

$$\boxed{F(x) = \int_a^x f(t)\, dt}$$

is a primitive of f, i.e. F is differentiable and $F'(x) = f(x)$.

Therefore for any primitive F of f:

$$\boxed{\int_a^b f(t)\, dt = F(b) - F(a)}$$

3.3 Integration by parts – Change of variable

3.3.1 Integration by parts

To find a primitive of a function f we can have the chance to recognize that f is the derivative of a well-known function. This is unfortunately very rarely the case, and we do not know the primitives of most functions. However we will see two techniques that allow us to calculate integrals and primitives: integration by parts and change of variable.

The integration by parts formula for primitives is:

$$\boxed{\int u(x)v'(x)\, dx = [uv] - \int u'(x)v(x)\, dx.}$$

and that of calculating definite integrals is

$$\boxed{\int_a^b u(x)v'(x)\, dx = [uv]_a^b - \int_a^b u'(x)v(x)\, dx.}$$

where $[uv] = u(x)v(x)$ and $[uv]_a^b = u(b)v(b) - u(a)v(a)$.
The use of integration by parts is based on the following idea: we do not know how to directly calculate the integral of a function f written like a product $f(x) = u(x)v'(x)$ but if we know how

to calculate the integral of $g(x) = u'(x)v(x)$ (which we hope is simpler) then by the formula of integration by parts we will get the integral of f.

Example 3.2 • **Calculation of** $\int \arcsin x\, dx$**.** To determine a primitive of $\arcsin x$, note that: $\arcsin x = 1 \cdot \arcsin x$. We set $u = \arcsin x$, $v' = 1$ (and therefore $u' = \frac{1}{\sqrt{1-x^2}}$ and $v = x$) then

$$\begin{aligned}\int 1 \cdot \arcsin x\, dx &= [x \arcsin x] - \int \frac{x}{\sqrt{1-x^2}}\, dx \\ &= x \arcsin x + \sqrt{1-x^2} + c\end{aligned}$$

• **Calculation of** $\int_0^1 x^2 \mathrm{e}^x\, dx$**.** Let $u = x^2$ and $v' = \mathrm{e}^x$ to get :

$$\int_0^1 x^2 \mathrm{e}^x\, dx = \left[x^2 \mathrm{e}^x\right]_0^1 - 2\int_0^1 x\mathrm{e}^x\, dx$$

We do a second integration by parts to calculate

$$\int_0^1 x\mathrm{e}^x\, dx = \left[x\mathrm{e}^x\right]_0^1 - \int_0^1 \mathrm{e}^x\, dx = 1$$

Then

$$\int_0^1 x^2 \mathrm{e}^x\, dx = \mathrm{e} - 2.$$

■

3.3.2 Change of variable

Let I and J be non-singular intervals. Let $u : I \to J$ and $F : J \to \mathbb{R}$ be differentiable functions. The function $F \circ u$ is then differentiable and $(F \circ u)' = u' \times F' \circ u$.
More lightly, this relation is denoted $(F(u))' = u'F'(u)$ and thus we can write

$$\int u'F'(u) = F(u) + C$$

Special cases:

- $\int u'u^n = \frac{1}{n+1}u^{n+1} + C$ for $n \in \mathbb{N}$.
- $\int \frac{u'}{u^n} = -\frac{1}{n-1}\frac{1}{u^{n-1}} + C$ for $n \in \mathbb{N}, n \geqslant 2$.
- $\int u'u^\alpha = \frac{1}{\alpha+1}u^{\alpha+1} + C$ for $\alpha \in \mathbb{R} - \{-1\}$
  and in particular $\int \frac{u'}{\sqrt{u}} = 2\sqrt{u} + C$.
- $\int \frac{u'}{u} = \ln|u| + C$.
- $\int u'\mathrm{e}^u = \mathrm{e}^u + C$.
- $\int u' \sin u = -\cos u + C$ and $\int u' \cos u = \sin u + C$
- $\int u' \cosh u = \sinh u + C$ and $\int u' \sinh u = \cosh u + C$.
- $\int \frac{u'}{1+u^2} = \arctan u + C$

- $\displaystyle\int \frac{\ln t}{t}\, dt = \int u' u = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln t)^2 + C.$
- $\displaystyle\int \frac{dt}{t \ln t} = \int \frac{u'}{u} = \ln|\ln t| + C.$
- $\displaystyle\int \tan t\, dt = \int \frac{\sin t}{\cos t}\, dt = \int -\frac{u'}{u} = -\ln|\cos t| + C.$
- $\displaystyle\int \frac{t}{1+t^2}\, dt = \int \frac{u'}{2u} = \frac{1}{2}\ln(1+t^2) + C.$
- $\displaystyle\int \frac{t}{1+t^4}\, dt = \int \frac{1}{2}\frac{u'}{1+u^2} = \frac{1}{2}\arctan t^2 + C.$

Let $u : I \to J$ derivable and $f : J \to \mathbb{R}$ having a primitive F.

To calculate $\displaystyle\int u'(t) f(u(t))\, dt$, we write $x = u(t)$, $dx = u'(t)dt$ and therefore

$$\int u'(t) f(u(t))\, dt = \int f(x)\, dx = F(x) + C = F(u(t)) + C$$

During this manipulation, we say that we have achieved the change of variable defined by the relation $x = u(t)$.

Example 3.3

- Let's calculate $\displaystyle\int \cos t \sin t^n\, dt$

  Let's change the variable $x = \sin t$ for which $dx = \cos t\, dt$. We obtain

$$\int \cos t \sin t^n\, dt = \int x^n\, dx = \frac{1}{n+1}x^{n+1} + C = \frac{1}{n+1}\sin^{n+1} t + C$$

- Let's calculate $\displaystyle\int \frac{dt}{\sqrt{t}+t}$ Let's change the variable $x = \sqrt{t}$ for which $t = x^2$ and $dt = 2x\, dx$.

$$\int \frac{dt}{\sqrt{t}+t} = \int \frac{2x\, dx}{x+x^2} = \int \frac{2\, dx}{1+x} = 2\ln|1+\sqrt{t}| + C$$

- Let's calculate $\displaystyle\int \frac{dt}{1+e^t}$

  Let's change the variable $x = e^t$ for which $dx = e^t\, dt = x\, dt$.

$$\int \frac{dt}{1+e^t} = \int \frac{dx}{x(1+x)} = \int \frac{1}{x} - \frac{1}{x+1} dx = \ln\frac{x}{x+1} + C = \ln\frac{e^t}{e^t+1} + C.$$

■

Theorem 3.3.1 Let $u : I \to J$ of class C^1 and $f : J \to \mathbb{R}$ continuous.
For all $a, b \in I$

$$\int_a^b f(u(t))\, u'(t)\, dt = \int_{u(a)}^{u(b)} f(x)\, dx$$

R To use this formula, we write :
$x = u(t)$, $dx = u'(t)dt$,
for $t = a$, $x = u(a)$,
for $t = b$, $x = u(b)$.

This allows us to formally transform one integral into the other.
We then say that we have achieved the change of variable defined by the relation $x = u(t)$.

■ **Example 3.4** • Let's calculate $\int_1^e \frac{dt}{t + t\ln t}$

We proceed to change the variable $x = \ln t$ for which $dx = \frac{dt}{t}$. for $t = 1$, $x = 0$ and for $t = e$, $x = 1$.
We obtain

$$\int_1^e \frac{dt}{t + t\ln t} = \int_0^1 \frac{dx}{1 + x} = \ln 2.$$

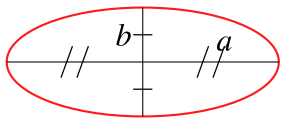
■

3.4 Integration of rational functions

Integration of rational fractions We know how to integrate a lot of simple functions. For example all polynomial functions: if $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ then $\int f(x)\,dx = a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + \cdots + a_n \frac{x^{n+1}}{n+1} + c$.

You should be aware, however, that many functions are not integrable using simple functions.

For example if $f(t) = \sqrt{a^2 \cos^2 t + b^2 \sin^2 t}$ then the integral $\int_0^{2\pi} f(t)\,dt$ can't be expressed as sum, product, inverse or composition of usual functions. In fact this integral is worth the length of an ellipse of parametric equation $(a\cos t, b\sin t)$; so there is no formula for the perimeter of an ellipse (unless $a = b$ in which case the ellipse is a circle!).



We first want to integrate the rational fractions

$$f(x) = \frac{\alpha x + \beta}{ax^2 + bx + c}$$

with $\alpha, \beta, a, b, c \in \mathbb{R}$, $a \neq 0$ and $(\alpha, \beta) \neq (0, 0)$.

First case. The denominator $ax^2 + bx + c$ has two distinct real roots $x_1, x_2 \in \mathbb{R}$.

Then $f(x)$ is also written $f(x) = \frac{\alpha x + \beta}{a(x - x_1)(x - x_2)}$ and there exist numbers $A, B \in \mathbb{R}$ such that $f(x) = \frac{A}{x - x_1} + \frac{B}{x - x_2}$. So we have

$$\int f(x)\,dx = A\ln|x - x_1| + B\ln|x - x_2| + c$$

on each of the intervals $]-\infty, x_1[$, $]x_1, x_2[$, $]x_2, +\infty[$ (if $x_1 < x_2$).

Let's determine $\int \frac{dx}{1-x^2}$ on $]-\infty,-1[,]-1,1[$ or $]1,+\infty[$.
We have the decomposition into simple elements

$$\frac{1}{1-x^2} = \frac{-1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{(A+B)x+A-B}{(x-1)(x+1)}$$

By identification, we find $A+B=0$ and $A-B=-1$.
Then $A=-\frac{1}{2}$ and $B=\frac{1}{2}$. As a result

$$\int \frac{dx}{1-x^2} = \frac{1}{2}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{dx}{x-1} = \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x-1| + C.$$

Second case. The denominator ax^2+bx+c has a double root $x_0 \in \mathbb{R}$.

Then $f(x) = \frac{\alpha x+\beta}{a(x-x_0)^2}$ and there exist numbers $A,B \in \mathbb{R}$ such as $f(x) = \frac{A}{(x-x_0)^2} + \frac{B}{x-x_0}$.
We then have

$$\int f(x)\,dx = -\frac{A}{x-x_0} + B\ln|x-x_0| + c$$

on each of the intervals $]-\infty,x_0[$, $]x_0,+\infty[$.

Third case. The denominator ax^2+bx+c does not have a real root. Let's see how to do it with an example.

Let $f(x) = \frac{x+1}{2x^2+x+1}$. Firstly, we show a fraction of the type $\frac{u'}{u}$ (which we know how to integrate into $\ln|u|$).

$$f(x) = \frac{(4x+1)\frac{1}{4} - \frac{1}{4} + 1}{2x^2+x+1} = \frac{1}{4}\cdot\frac{4x+1}{2x^2+x+1} + \frac{3}{4}\cdot\frac{1}{2x^2+x+1}$$

We can integrate the fraction $\frac{4x+1}{2x^2+x+1}$:

$$\int \frac{4x+1}{2x^2+x+1}\,dx = \int \frac{u'(x)}{u(x)}\,dx = \ln\left|2x^2+x+1\right| + c$$

Let's take care of the other part $\frac{1}{2x^2+x+1}$, we will write it in the form $\frac{1}{u^2+1}$ (admiting as primitive $\arctan u$).

$$\begin{aligned}
\frac{1}{2x^2+x+1} &= \frac{1}{2}\frac{1}{(x+\frac{1}{4})^2 - \frac{1}{16} + \frac{1}{2}} = \frac{1}{2(x+\frac{1}{4})^2 + \frac{7}{8}} \\
&= \frac{8}{7}\cdot\frac{1}{\frac{8}{7}2(x+\frac{1}{4})^2+1} = \frac{8}{7}\cdot\frac{1}{\left(\frac{4}{\sqrt{7}}(x+\frac{1}{4})\right)^2+1}
\end{aligned}$$

We pose the change of variable $u = \frac{4}{\sqrt{7}}(x+\frac{1}{4})$ (and so $du = \frac{4}{\sqrt{7}}dx$) to find

$$\begin{aligned}\int \frac{dx}{2x^2+x+1} &= \int \frac{8}{7}\frac{dx}{\left(\frac{4}{\sqrt{7}}(x+\frac{1}{4})\right)^2+1} = \frac{8}{7}\int \frac{du}{u^2+1}\cdot\frac{\sqrt{7}}{4} \\ &= \frac{2}{\sqrt{7}}\arctan u + c = \frac{2}{\sqrt{7}}\arctan\left(\frac{4}{\sqrt{7}}(x+\frac{1}{4})\right)+c\,.\end{aligned}$$

Eventually :

$$\int f(x)\,dx = \frac{1}{4}\ln\left(2x^2+x+1\right)+\frac{3}{2\sqrt{7}}\arctan\left(\frac{4}{\sqrt{7}}(x+\frac{1}{4})\right)+c$$

Let $\frac{P(x)}{Q(x)}$ be a rational fraction, where $P(x), Q(x)$ are polynomials with real coefficients. Then the fraction $\frac{P(x)}{Q(x)}$ is written as the sum of a polynomial $E(x) \in \mathbb{R}[x]$ and simple elements of one of the following forms:

$$\frac{\gamma}{(x-x_0)^k} \quad \text{or} \quad \frac{\alpha x+\beta}{(ax^2+bx+c)^k} \text{ with } b^2-4ac<0$$

where $\alpha, \beta, \gamma, a, b, c \in \mathbb{R}$ and $k \in \mathbb{N}\setminus\{0\}$.

- We know how to integrate the polynomial $E(x)$.
- Integration of the simple element $\frac{\gamma}{(x-x_0)^k}$.
  - if $k=1$ then $\int \frac{\gamma\,dx}{x-x_0} = \gamma\ln|x-x_0|+c_0$ (on $]-\infty, x_0[$ or $]x_0,+\infty[$).
- – if $k\geq 2$ then $\int \frac{\gamma\,dx}{(x-x_0)^k} = \gamma\int (x-x_0)^{-k}\,dx = \frac{\gamma}{-k+1}(x-x_0)^{-k+1}+c_0$ (on $]-\infty, x_0[$ or $]x_0,+\infty[$).
- integration of the Simple element $\frac{\alpha x+\beta}{(ax^2+bx+c)^k}$. We write this fraction in the form

$$\frac{\alpha x+\beta}{(ax^2+bx+c)^k} = \gamma\frac{2ax+b}{(ax^2+bx+c)^k}+\delta\frac{1}{(ax^2+bx+c)^k}$$

and we proceed according to the value of the parameter k as follows.

1. If $k=1$, we have $\int \frac{2ax+b}{ax^2+bx+c}\,dx = \int \frac{u'(x)}{u(x)}\,dx = \ln|u(x)|+c_0 = \ln|ax^2+bx+c|+c_0$.
2. If $k\geq 2$, then $\int \frac{2ax+b}{(ax^2+bx+c)^k}\,dx = \int \frac{u'(x)}{u(x)^k}\,dx = \frac{1}{-k+1}u(x)^{-k+1}+c_0 = \frac{1}{-k+1}(ax^2+bx+c)^{-k+1}+c_0$.
3. if $k=1$, then $\int \frac{1}{ax^2+bx+c}\,dx$. By changing the variable $u=px+q$ we go back to calculate a primitive of the type $\int \frac{du}{u^2+1} = \arctan u + c_0$.

4. If $k \geq 2$, calculating $\displaystyle\int \frac{1}{(ax^2+bx+c)^k}\,dx$, We proceed to the change of variable $u = px+q$ to reduce the calculation into that of $I_k = \displaystyle\int \frac{du}{(u^2+1)^k}$. An integration by parts allows us to go from I_k to I_{k-1}.

3.5 Integration of trigonometric functions

We can also calculate the primitives of the form $\displaystyle\int P(\cos x, \sin x)\,dx$ or of the form $\displaystyle\int \frac{P(\cos x, \sin x)}{Q(\cos x, \sin x)}\,dx$ when P and Q are polynomials, reducing to integrating a rational fraction.

There are two methods:

- Bioche's rules are quite effective but won't always work;
- changing variable $t = \tan\frac{x}{2}$ works all the time but leads to more calculations.

We note $\omega(x) = f(x)\,dx$. We then have $\omega(-x) = f(-x)\,d(-x) = -f(-x)\,dx$ and $\omega(\pi-x) = f(\pi-x)\,d(\pi-x) = -f(\pi-x)\,dx$.

- If $\omega(-x) = \omega(x)$ then we proceed to the change of variable $u = \cos x$.
- if $\omega(\pi-x) = \omega(x)$ then we proceed to the change of variable $u = \sin x$.
- If $\omega(\pi+x) = \omega(x)$ then we proceed to the change of variable $u = \tan x$.

Example 3.5 Calculation of the primitive $\displaystyle\int \frac{\cos x\,dx}{2-\cos^2 x}$.

We note

$$\omega(x) = \frac{\cos x\,dx}{2-\cos^2 x}.$$

Since

$$\omega(\pi-x) = \frac{\cos(\pi-x)\,d(\pi-x)}{2-\cos^2(\pi-x)} = \frac{(-\cos x)\,(-dx)}{2-\cos^2 x} = \omega(x)$$

then the appropriate variable change is $u = \sin x$ for which $du = \cos x\,dx$. So:

$$\begin{aligned} \int \frac{\cos x\,dx}{2-\cos^2 x} &= \int \frac{\cos x\,dx}{2-(1-\sin^2 x)} \\ &= \int \frac{du}{1+u^2} = \left[\arctan u\right] \\ &= \arctan(\sin x) + c\,. \end{aligned}$$

This method allows us to express sin, cosine and tangent in terms of $\tan\frac{x}{2}$.

$$\text{with} \quad t = \tan\frac{x}{2} \quad \text{we have}$$

$$\cos x = \frac{1-t^2}{1+t^2} \qquad \sin x = \frac{2t}{1+t^2} \qquad \tan x = \frac{2t}{1-t^2}$$

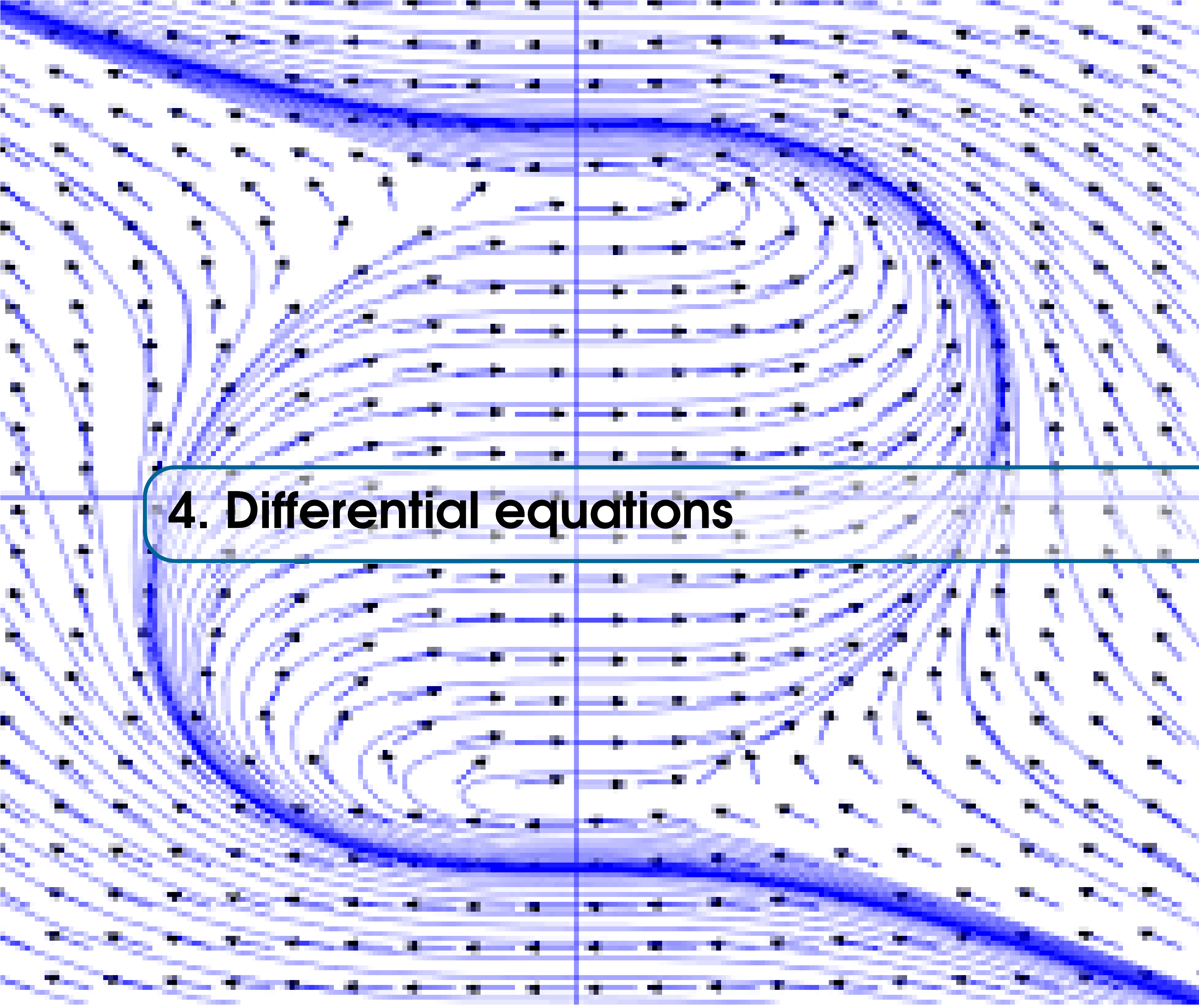
$$\text{and} \qquad dx = \frac{2\,dt}{1+t^2}.$$

Calculation of the integral $\displaystyle\int_{-\pi/2}^{0} \frac{dx}{1-\sin x}$.

The change of variable $t = \tan\frac{x}{2}$ defines a bijection from $[-\frac{\pi}{2}, 0]$ into $[-1, 0]$ (for $x = -\frac{\pi}{2}$, $t = -1$ and for $x = 0$, $t = 0$). In addition we have $\sin x = \frac{2t}{1+t^2}$ and $dx = \frac{2\,dt}{1+t^2}$.

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{0} \frac{dx}{1-\sin x} &= \int_{-1}^{0} \frac{\frac{2\,dt}{1+t^2}}{1-\frac{2t}{1+t^2}} = 2\int_{-1}^{0} \frac{dt}{1+t^2-2t} \\
&= 2\int_{-1}^{0} \frac{dt}{(1-t)^2} = 2\left[\frac{1}{1-t}\right]_{-1}^{0} = 2\big(1-\frac{1}{2}\big) = 1
\end{aligned}$$

■

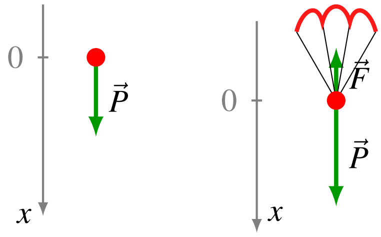


4. Differential equations

When a body falls in free fall without friction, it is only subject to its weight $\vec{P}$. By the fundamental principle of mechanics: $\vec{P} = m\vec{a}$. All vectors are vertical so $mg = ma$, where g is the gravitational constant, a the vertical acceleration and m the mass. We obtain $a = g$. The acceleration being the derivative of the speed with respect to time, we obtain:

$$\frac{v(t)}{t} = g$$

It is easy to deduce the speed by integration: $v(t) = gt$ (assuming initial velocity is zero), i.e. that the speed increases linearly over time. Since the speed is the derivative of the position, we have $v(t) = \frac{x(t)}{t}$, so by a new integration we obtain $x(t) = \frac{1}{2}gt^2$ (assuming the initial position is zero).



The case of a parachutist is more complicated. The previous model is not applicable because it does not take friction into account. The parachute makes undergo a friction force opposite to its speed. We assume that the friction is proportional to the speed: $F = -fmv$ (f is the coefficient of friction). Thus the fundamental principle of mechanics becomes $mg - fmv = ma$, which leads to the relationship :

$$\frac{v(t)}{t} = g - fv(t)$$

It is a relationship between the speed v and its derivative: it is a *differential equation*. It is not easy to find which function v is appropriate.

GENERALITIES

4.1 Generalities

A differential equation is an equation:

- whose unknown is a function (generally denoted $y(x)$ or simply y);
- in which appears some of the derivatives of the function (first derivative y', or higher order derivatives y'', $y^{(3)}, \ldots$).

■ **Example 4.1 — Easy to Solve Differential Equations.** Find at least one function, solution of following differential equations:

$y' = \sin x$ y(x) = -cos$x + k$ where $k \in \mathbb{R}$
y' = 1 + e^x y(x) = x + e$^x + k$ where $k \in \mathbb{R}$

$y' = y$ y(x) = k e^x where $k \in \mathbb{R}$
y' = 3y $y(x) = ke^{3x}$ where $k \in \mathbb{R}$
y" = cos$x$ y(x) = -cos$x + ax + b$ where $a, b \in \mathbb{R}$
y" = y $y(x) = ae^x + be^{-x}$ where $a, b \in \mathbb{R}$

■

Definition 4.1.1

- A **differential equation** of order n is an equation of the form

$$F\left(x, y, y', \ldots, y^{(n)}\right) = 0 \qquad (E)$$

where F is a function of $(n+2)$ variables.

- A **solution** of such an equation on an interval $I \subset \mathbb{R}$ is a function $y : I \to \mathbb{R}$ which is n

times differentiable and which verifies the equation (E).

Example 4.2 1. Let the differential equation $y' = 2xy + 4x$. Check that $y(x) = k\exp(x^2) - 2$ is a solution on $\mathbb{R}$, this for all $k \in \mathbb{R}$.

2. Consider the differential equation $x^2y'' - 2y + 2x = 0$. Check that $y(x) = kx^2 + x$ is a solution on $\mathbb{R}$, for all $k \in \mathbb{R}$.

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- It is customary for differential equations to write y instead of $y(x)$, y' instead of $y'(x)$,... We therefore note «$y' = \sin x$ » which means «$y'(x) = \sin x$ ».
- You have to get used to changing names for functions and variables. For example $(x'')^3 + t(x')^3 + (\sin t)x^4 = e^t$ is a differential equation of order 2, whose unknown is a function x which depends on the variable t. We are therefore looking for a function $x(t)$, twice differentiable, which satisfies $(x''(t))^3 + t(x'(t))^3 + (\sin t)(x(t))^4 = e^t$.
- Searching for a primitive already means solving the differential equation $y' = f(x)$. This is why we often find «integrating the differential equation » to «find the solutions of the differential equation ».
- The notion of interval in solving a differential equation is fundamental. If we change the interval, we can very well obtain other solutions.

Definition 4.1.2 A differential equation **with separate variables** is an equation of the type:

$$y' = g(x)/f(y) \qquad \text{or} \qquad y'f(y) = g(x)$$

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Such an equation is solved by calculating primitives. If $G(x)$ is a primitive of $g(x)$ then $G'(x) = g(x)$. If $F(x)$ is a primitive of $f(x)$ then $F'(x) = f(x)$, but above all, by derivation of a composition, $\big(F(y(x))\big)' = y'(x)F'(y(x)) = y'f(y)$. Thus the differential equation $y'f(y) = g(x)$ is rewritten $\big(F(y(x))\big)' = G'(x)$ which is equivalent to an equality of functions: $F(y(x)) = G(x) + c$.

Example 4.3 Here is a concrete example:

$$x^2y' = e^{-y}$$

We start by separating the variables x on one side and y on the other: $y'e^y = \frac{1}{x^2}$ (assuming $x \neq 0$). We integrate on both sides:

$$e^y = -\frac{1}{x} + c \quad (c \in \mathbb{R})$$

Which allows us to obtain y (assuming $-\frac{1}{x} + c > 0$):

$$y(x) = \ln\left(-\frac{1}{x} + c\right)$$

which is a solution on each interval I where it is defined and differentiable. This interval depends on the constant c : if $c<0$, $I=]\frac{1}{c},0[$; if $c=0$, $I=]-\infty,0[$; if $c>0$, $I=]\frac{1}{c},+\infty[\cup]-\infty,0[$. ■

4.1.1 Linear differential equation

Definition 4.1.3

- A differential equation of order n is **linear** if it is of the form

$$a_0(x)y+a_1(x)y'+\cdots+a_n(x)y^{(n)}=g(x)$$

where the a_i and g are real continuous functions on an interval $I\subset\mathbb{R}$.
The term linear roughly means that there is no exponent for the terms $y,y',y'',\ldots$

- A linear differential equation is **homogeneous**, or **without second member**, if the function g above is the null function:

$$a_0(x)y+a_1(x)y'+\cdots+a_n(x)y^{(n)}=0$$

- A linear differential equation is **with constant coefficients** if the a_i functions above are constant:

$$a_0y+a_1y'+\cdots+a_ny^{(n)}=g(x)$$

where the a_i are real constants and g a continuous function.

■ **Example 4.4**

1. $y'+5xy=e^x$ is a first order linear differential equation with right side.
2. $y'+5xy=0$ is the homogeneous differential equation associated with the previous one.
3. $2y''-3y'+5y=0$ is a second order linear differential equation with constant coefficients, without a second member.
4. $y'^2-y=x$ or $y''\cdot y'-y=0$ are not linear differential equations.

■

Proposition 4.1.1 — Principle of linearity. if y_1 and y_2 are solutions of the homogeneous linear differential equation

$$a_0(x)y+a_1(x)y'+\cdots+a_n(x)y^{(n)}=0 \qquad (E_0)$$

then, for every $\lambda,\mu\in\mathbb{R}$, $\lambda y_1+\mu y_2$ is also solution of this equation.

(R) It's a simple check. We can reformulate the proposition by saying that the set of solutions forms a vector space.

To solve a linear differential equation with right hand side

$$a_0(x)y+a_1(x)y'+\cdots+a_n(x)y^{(n)}=g(x), \qquad (E)$$

We often break down the resolution into two steps:

- find a particular solution y_0 of the equation (E),
- find the set $\mathscr{S}_h$ of solutions y of the associated homogeneous equation

$$a_0(x)y+a_1(x)y'+\cdots+a_n(x)y^{(n)}=0 \qquad (E_0)$$

which makes it possible to find all the solutions(E)

Proposition 4.1.2 — Principle of superposition. The set of all solutions $\mathscr{S}$ of (E) is formed of

$$y_0 + y \quad \text{with} \quad y \in \mathscr{S}_h.$$

In other words, we find all the solutions by adding a particular solution to the solutions of the homogeneous equation. This is an immediate consequence of the linear nature of the equations.

FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS

4.2 First order linear differential equations

Definition 4.2.1 A **first order linear** differential equation is an equation of the type :

$$y' = a(x)y + b(x) \tag{E}$$

where a and b are functions defined on an open interval I of $\mathbb{R}$.

In the following we will assume that a and b are continuous functions on I. We can consider the form : $\alpha(x)y' + \beta(x)y = \gamma(x)$. We will then ask that $\alpha(x) \neq 0$ for all $x \in I$. Division by α allows us to find the form (E).

We will start by solving the case where a is a constant and $b = 0$. Then a will be a function (and always $b = 0$). We will end with the general case where a and b are two functions.

Theorem 4.2.1 Let a be a real number. Let the differential equation be :

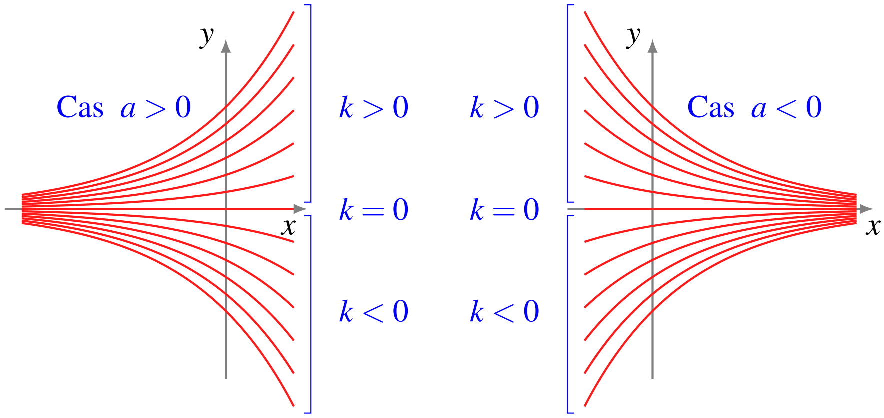
$$y' = ay \tag{E}$$

The solutions of (E), on $\mathbb{R}$, are the functions y defined by:

$$\boxed{y(x) = ke^{ax}}$$

where $k \in \mathbb{R}$ is any constant.

Explanation : rewriting the differential equation in the form $\boxed{\frac{y'}{y} = a}$ and after integration, we find : $\boxed{\ln|y(x)| = ax + b}$. By the exponential on both sides: $\boxed{|y(x)| = e^{ax+b}}$ i.e : $y(x) = \pm e^b e^{ax}$. Let $k = \pm e^b$ to get the required form of the solution.



■ **Example 4.5** Solve the differential equation:

$$3y' - 5y = 0$$

We write this equation in the form $y' = \frac{5}{3}y$. Its solutions, on $\mathbb{R}$, are therefore of the form: $y(x) = ke^{\frac{5}{3}x}$, where $k \in \mathbb{R}$. ■

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- The differential equation (E) admits an infinity of solutions (since we have an infinite choice of the constant k).
- The constant k can be zero. In this case, we obtain the «null solution »: $y = 0$ on $\mathbb{R}$, which is a trivial solution from the differential equation.
- The theorem 4.2.1 can also be interpreted as follows: if y_0 is a non null solution of the differential equation (E), then all other solutions y are multiples of y_0.

The following theorem states that, when a is a function, solve the differential equation $y' = a(x)y$ amounts to determining a primitive A of a (which is not always explicitly possible).

Theorem 4.2.2 Let $a : I \to \mathbb{R}$ be a continuous function. Let $A : I \to \mathbb{R}$ be a primitive of a. Let the differential equation be :

$$y' = a(x)y \qquad (E)$$

The solutions on I of (E) are the functions y defined by :

$$\boxed{y(x) = ke^{A(x)}}$$

where $k \in \mathbb{R}$ is any constant.

If $a(x) = a$ is a constant function, then a primitive is for example $A(x) = ax$ and we find the solutions of theorem 4.2.1.

A quick proof of the theorem 4.2.2 is as follows :

$$\begin{aligned} & \frac{y'}{y} = a(x) \iff \ln|y(x)| = A(x) + b \iff |y(x)| = e^{A(x)+b} \\ \iff \quad & y(x) = \pm e^b e^{A(x)} \iff y(x) = ke^{A(x)} \quad \text{with } k = \pm e^b \end{aligned}$$

A rigorous proof (since we avoid dividing by something that could be null):

$$\begin{aligned} y(x) \text{ solution of } (E) \quad & : y'(x) - a(x)y(x) = 0 \\ \iff \quad & e^{-A(x)}\big(y'(x) - ay(x)\big) = 0 \\ \iff \quad & \big(y(x)e^{-A(x)}\big)' = 0 \\ \iff \quad & \exists k \in \mathbb{R} \quad y(x)e^{-A(x)} = k \\ \iff \quad & \exists k \in \mathbb{R} \quad y(x) = ke^{A(x)} \end{aligned}$$

■ **Example 4.6** How to solve the differential equation $x^2y' = y$? We place ourselves on the interval $I_+ =]0, +\infty[$ or $I_- =]-\infty, 0[$. The equation becomes

$$y' = \frac{1}{x^2}y.$$

So

$$a(x) = \frac{1}{x^2},$$

of which a primitive is

$$A(x) = -\frac{1}{x}.$$

So the solutions are

$$y(x) = ke^{-\frac{1}{x}},$$

where $k \in \mathbb{R}$. ■

We are left with the general case of the linear differential equation of order 1 with second member :

$$y' = a(x)y + b(x) \qquad (E)$$

where $a : I \to \mathbb{R}$ and $b : I \to \mathbb{R}$ are continuous functions.

The associated homogeneous equation is:

$$y' = a(x)y \qquad (E_0)$$

There **is no** new formula to learn for this case. It is enough to apply the **principle of superposition**: the solutions of (E) are obtained by adding to a particular solution of (E) the solutions of (E_0).

Proposition 4.2.3 If y_0 is a solution of (E), then the solutions of (E) are the functions $y : I \to \mathbb{R}$ defined by:

$$y(x) = y_0(x) + ke^{A(x)} \qquad \text{with } k \in \mathbb{R}$$

where $x \mapsto A(x)$ s a primitive of $x \mapsto a(x)$.

R The search for the general solution of (E) is reduced to looking for a particular solution. Sometimes this is done by noticing an obvious solution.

■ **Example 4.7** For example, the differential equation $y' = 2xy + 4x$ has the particular solution $y_0(x) = -2$; therefore the set of solutions of this equation are the $y(x) = -2 + ke^{x^2}$, where $k \in \mathbb{R}$.

The general solution of (E_0) $y' = a(x)y$ is given by $y(x) = ke^{A(x)}$, with $k \in \mathbb{R}$ a constant. ■

Proposition 4.2.4 The **method of variation of constants** consists of looking for a particular solution in the form $y_0(x) = k(x)e^{A(x)}$, where k is now a function to be determined so that y_0 is a solution of (E) $y' = a(x)y + b(x)$.

$$y_0'(x) = a(x)k(x)e^{A(x)} + k'(x)e^{A(x)} = a(x)y_0(x) + k'(x)e^{A(x)}$$

Therefore $\boxed{y_0'(x) - a(x)y_0(x) = k'(x)e^{A(x)}}$.
Then y_0 is a solution of (E) if and only if

$$k'(x)e^{A(x)} = b(x) \iff k'(x) = b(x)e^{-A(x)} \iff k(x) = \int b(x)e^{-A(x)}x.$$

which gives a particular solution $y_0(x) = \left(\int b(x)e^{-A(x)}x\right)e^{A(x)}$ of (E) on I. The general solution of (E) is given by

$$y(x) = y_0(x) + ke^{A(x)}, \quad k \in \mathbb{R}.$$

■ **Example 4.8** Let the equation $y' + y = e^x + 1$. The homogeneous equation is $y' = -y$ whose solutions are the $y(x) = ke^{-x}$, $k \in \mathbb{R}$.

Let's look for a particular solution with the method of variation of constants : we note $y_0(x) = k(x)e^{-x}$. We must find $k(x)$ so that y_0 verifies the equation differential $y' + y = e^x + 1$.

$$\begin{aligned} y_0' + y_0 = e^x + 1 &\iff \left(k'(x)e^{-x} - k(x)e^{-x}\right) + k(x)e^{-x} = e^x + 1 \\ &\iff k'(x)e^{-x} = e^x + 1 \\ &\iff k'(x) = e^{2x} + e^x \iff k(x) = \frac{1}{2}e^{2x} + e^x + c \end{aligned}$$

We set $c = 0$ (any value is suitable):

$$y_0(x) = k(x)e^{-x} = \left(\frac{1}{2}e^{2x} + e^x\right)e^{-x} = \frac{1}{2}e^x + 1$$

We have our special solution! General solutions of the equation $y' + y = e^x + 1$ are therefore:

$$y(x) = \frac{1}{2}e^x + 1 + ke^{-x}, \qquad k \in \mathbb{R}.$$

■

Specific Differential Equations

4.2.1 Specific Differential Equations

Definition 4.2.2 — Bernoulli's equation

. It concerns differential equations of the form

$$y'(x) = a(x)y + b(x)y^{\alpha},\ \alpha \in \mathbb{R}\backslash\{0,1\}.$$

We seek solutions that do not vanish

$$\frac{y'}{y^{\alpha}} = \frac{a(x)}{y^{\alpha-1}} + b(x).$$

We set $z = y^{1-\alpha}$. We obtain

$$\frac{1}{1-\alpha}z' = a(x)z + b(x).$$

We obtain a first-order linear equation in z, which we know how to solve.

Example 4.9 So, to solve

$$y' = y^3 - \frac{y}{x}.$$

We set $z = \frac{1}{y^2}$. Therefore, we obtain

$$-\frac{z'}{2} = 1 - \frac{z}{x} \implies z(x) = 2x + \lambda x^2,$$

which ultimately yields

$$y(x) = \pm\frac{1}{\sqrt{2x+\lambda x^2}}.$$

■

Definition 4.2.3 — Riccati's equation. It concerns differential equations of the form

$$y'(x) = a(x)y^2 + b(x)y + c(x).$$

If a particular solution y_0 is known, then we can solve this differential equation. Indeed, we set $y = y_0 + z$ and by substitution, we find

$$z' = (2a(x)y_0(x) + b(x))z + a(x)z^2.$$

Thus, we obtain a Bernoulli equation, which we can solve.

R

The term **homogeneous differential equation** has two entirely distinct and independent meanings. (One already seen previously)

Definition 4.2.4 — First-order homogeneous differential equation of degree n. A first-order but not necessarily linear differential equation is said to be homogeneous of degree n if it can be written in the form:

$$\frac{dy}{dx} = x^n h\left(\frac{y}{x}\right).$$

The most studied case is when $n = 0$, and solving such an equation, using the substitution $u(x) = \frac{y(x)}{x}$, transforms the equation $\frac{dy}{dx} = h\left(\frac{y}{x}\right)$ into a separable variable equation:

$$\frac{u'(x)}{h(u(x)) - u(x)} = \frac{1}{x}.$$

SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

4.3 Second order linear differential equation with constant coefficients

Definition 4.3.1 A second order linear differential equation with constant coefficients, is an equation of the form

$$ay'' + by' + cy = g(x) \tag{E}$$

where $a, b, c \in \mathbb{R}$, $a \neq 0$ and $g$is a continuous function on an open interval I.

The equation

$$ay'' + by' + cy = 0 \tag{E_0}$$

is called the homogeneous equation associated with (E).

4.3.1 The homogeneous equation case

The structure of solutions to the equation is very simple:

Theorem 4.3.1 The set of solutions of the homogeneous equation (E_0) is an $\mathbb{R}$-vectorial space of dimension 2.

We are looking for a solution to (E_0) Under the form $y(x) = e^{rx}$ where $r \in \mathbb{C}$ is a constant to be determined. We find

$$\begin{aligned} & ay'' + by' + cy = 0 \\ \iff & (ar^2 + br + c)e^{rx} = 0 \\ \iff & ar^2 + br + c = 0. \end{aligned}$$

Definition 4.3.2 The equation $ar^2 + br + c = 0$ is called **the characteristic equation** associated with (E_0).

Let $\Delta = b^2 - 4ac$, the discriminant of the characteristic equation associated with(E_0).

Theorem 4.3.2 1. If $\Delta > 0$, the characteristic equation has two distinct real roots $r_1 \neq r_2$ and the solutions of (E_0) are

$$\boxed{y(x) = \lambda e^{r_1 x} + \mu e^{r_2 x} \quad \text{where } \lambda, \mu \in \mathbb{R}.}$$

2. If $\Delta = 0$, the characteristic equation has a double root r_0 and the solutions of (E_0) are

$$y(x) = (\lambda + \mu x)e^{r_0 x} \quad \text{where } \lambda, \mu \in \mathbb{R}.$$

3. If $\Delta < 0$, the characteristic equation has two complex roots $r_1 = \alpha + \beta$, $r_2 = \alpha - \beta$ and the solutions of (E_0) are

$$y(x) = e^{\alpha x}\big(\lambda \cos(\beta x) + \mu \sin(\beta x)\big) \quad \text{where } \lambda, \mu \in \mathbb{R}.$$

Example 4.10

1. Solve on $\mathbb{R}$ the equation $y'' - y' - 2y = 0$.
   The characteristic equation is $r^2 - r - 2 = 0$, which is also written $(r+1)(r-2) = 0$ $(\Delta > 0)$. Hence, for every $x \in \mathbb{R}$, $y(x) = \lambda e^{-x} + \mu e^{2x}$, with $\lambda, \mu \in \mathbb{R}$.
2. Solve on $\mathbb{R}$ the equation $y'' - 4y' + 4y = 0$.
   The characteristic equation is $r^2 - 4r + 4 = 0$, let $(r-2)^2 = 0$ $(\Delta = 0)$. Therefore, for all $x \in \mathbb{R}$, $y(x) = (\lambda x + \mu)e^{2x}$, with $\lambda, \mu \in \mathbb{R}$.
3. Solve on $\mathbb{R}$ the equation $y'' - 2y' + 5y = 0$.
   The characteristic equation is $r^2 - 2r + 5 = 0$. It admits two complex solutions: $r_1 = 1 + 2$ and $r_2 = 1 - 2$ $(\Delta < 0)$. Therefore, for all $x \in \mathbb{R}$, $y(x) = e^x(\lambda \cos(2x) + \mu \sin(2x))$, with $\lambda, \mu \in \mathbb{R}$.

■

4.3.2 Equation with second member

We move on to the general case of a linear differential equation of order 2, with constant coefficients, but with a second member g which is a continuous function on an open interval $I \subset \mathbb{R}$:

$$ay'' + by' + cy = g(x) \qquad (E)$$

For this type of equation, we admit the following theorem: [Cauchy-Lipschitz theorem] For every $x_0 \in I$ and any couple $(y_0, y_1) \in \mathbb{R}^2$, the equation (E) admits a *unique* solution y on I satisfying the initial conditions:

$$y(x_0) = y_0 \quad \text{and} \quad y'(x_0) = y_1.$$

Proposition 4.3.3 — Principle of superposition. General solutions of the equation (E) are obtained by adding the general solutions of the homogeneous equation (E_0) to a particular solution of (E).

Second member of the type $e^{\alpha x}P(x)$.

If $g(x) = e^{\alpha x}P(x)$, with $\alpha \in \mathbb{R}$ and $P \in \mathbb{R}[X]$, then we look for a particular solution in the form $y_0(x) = e^{\alpha x}x^m Q(x)$, where Q is a polynomial of same degree as P with:

- $y_0(x) = e^{\alpha x}Q(x)$ $(m = 0)$, if α is not a root of the characteristic equation,
- $y_0(x) = xe^{\alpha x}Q(x)$ $(m = 1)$, if α is a simple root of the characteristic equation,
- $y_0(x) = x^2 e^{\alpha x}Q(x)$ $(m = 2)$, if α is a double root of the characteristic equation.

Second member of the type $e^{\alpha x}\big(P_1(x)\cos(\beta x) + P_2(x)\sin(\beta x)\big)$.

If $g(x) = e^{\alpha x}\big(P_1(x)\cos(\beta x) + P_2(x)\sin(\beta x)\big)$, where $\alpha, \beta \in \mathbb{R}$ and $P_1, P_2 \in \mathbb{R}[X]$, we seek a particular solution in the form :

- $y_0(x) = e^{\alpha x}\big(Q_1(x)\cos(\beta x) + Q_2(x)\sin(\beta x)\big)$, if $\alpha + \beta$ is not a root of the characteristic equation,
- $y_0(x) = xe^{\alpha x}\big(Q_1(x)\cos(\beta x) + Q_2(x)\sin(\beta x)\big)$, if $\alpha + \beta$ is a root of the characteristic equation.

In both cases, Q_1 and Q_2 are two polynomials of degree $n = \max\{\deg P_1, \deg P_2\}$.

Example 4.11 Solve differential equations :

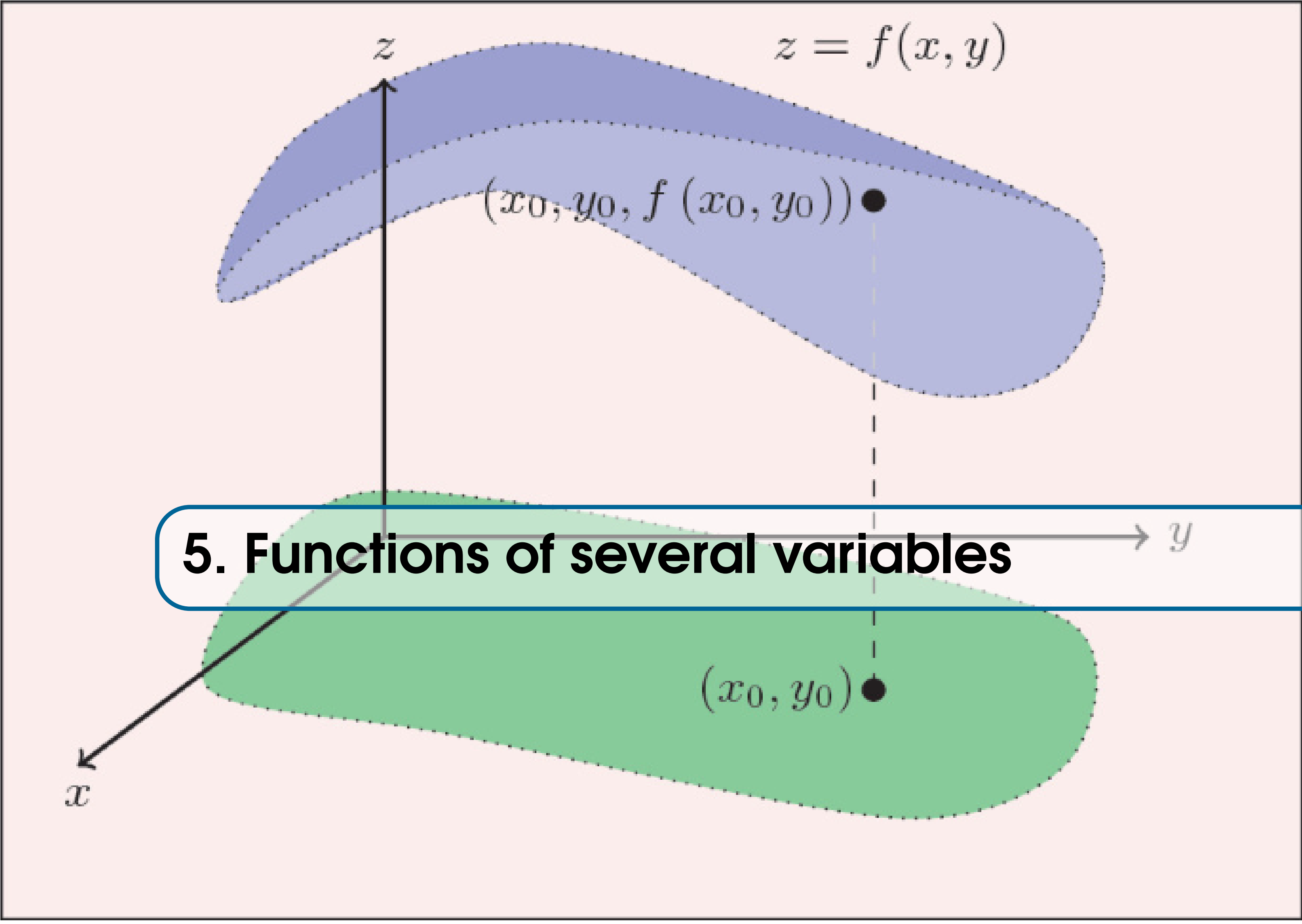
$$(E_0)\ y'' - 5y' + 6y = 0 \quad (E_1)\ y'' - 5y' + 6y = 4xe^x$$

1. **Equation** (E_0)**.** The characteristic equation is $r^2 - 5r + 6 = (r-2)(r-3) = 0$, with two distinct roots $r_1 = 2, r_2 = 3$. Therefore, the set of all solutions of (E_0) is $\big\{\lambda e^{2x} + \mu e^{3x} \mid \lambda, \mu \in \mathbb{R}\big\}$.
2. **Equation** (E_1)**.**
   (a) We seek a particular solution to (E_1) in the form $y_0(x) = (ax+b)e^x$. When we inject y_0 into the equation (E_1), we obtain :

$$\begin{aligned} & (ax+2a+b)e^x - 5(ax+a+b)e^x + 6(ax+b)e^x = 4xe^x \\ \Longleftrightarrow\ & (a - 5a + 6a)x + 2a + b - 5(a+b) + 6b = 4x \\ \Longleftrightarrow\ & 2a = 4 \ \text{ and } \ -3a + 2b = 0 \quad (a = 2 \ \text{ and } \ b = 3) \end{aligned}$$

   Then $y_0(x) = (2x+3)e^x$.
   (b) The set of solutions is $\big\{(2x+3)e^x + \lambda e^{2x} + \mu e^{3x} \mid \lambda, \mu \in \mathbb{R}\big\}$.

■



5. Functions of several variables

In this chapter, we present the fundamental concepts of the analysis of functions of several variables.
A numerical function of several real variables is a function whose domain E is a subset of $\mathbb{R}^n$. The codomain F can be $\mathbb{R}$ or $\mathbb{R}^p$. The second case can be reduced to the first case by considering that it actually consists of p functions from $\mathbb{R}^n$ to $\mathbb{R}$ called coordinate functions.

To model many phenomena, single-variable functions are not sufficient; we often need functions of several variables.

■ **Example 5.1** For a sample of a mole of Van der Waals gas, the pressure P of the gas is a function of two variables: its temperature T, and the volume V occupied by this sample. Indeed, we have:

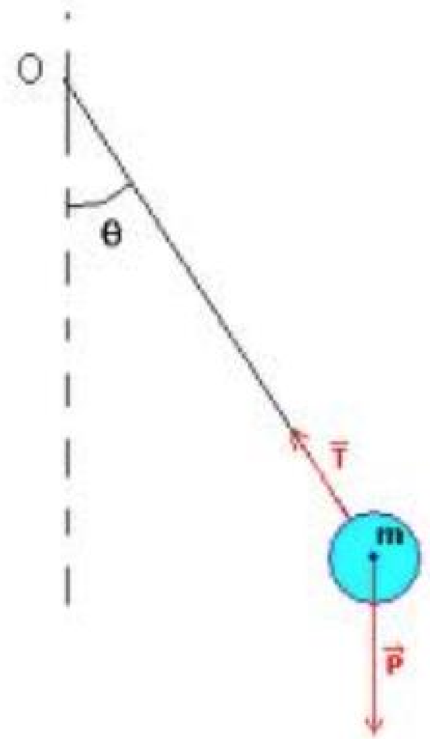
$$P(T,V) = \frac{RT}{V-b} - \frac{a}{V^2}$$

where a, b, and R are constants (a and b depend on the gas considered, R is a universal constant).
■

■ **Example 5.2** The total energy E of a pendulum is a function of two variables: the angle θ that the pendulum makes with the vertical, and its angular velocity $\dot{\theta}$. Indeed, we have:

$$E(\theta,\dot{\theta}) = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell(1-\cos\theta)$$

where m, g, and ℓ are constants (the mass of the pendulum, the universal gravitational constant, and the length of the pendulum's rod).



■

Definition, Limit, Continuity, and Partial Derivatives

5.1 Definition, Limit, Continuity, and Partial Derivatives

5.1.1 Functions of Two Variables with Real Values

Definition 5.1.1 Let D be a subset of $\mathbb{R}^2$, meaning a set of pairs of real numbers (x,y).
A function of two variables defined on D is the process of associating a unique real number to each pair (x,y) in D. It is generally denoted as $f(x,y) = z$.

R Just as for single-variable functions, we can talk about bounded, upper-bounded, lower-bounded functions of two variables, and we can also talk about extrema of a function of two variables.
However, there is no defined notion of monotonicity for functions of two variables.

Example 5.3 Let $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ be the rational function defined by

$$f(x,y) = \frac{xy}{x^2+y^2}$$

The function f is bounded by $1/2$. ∎

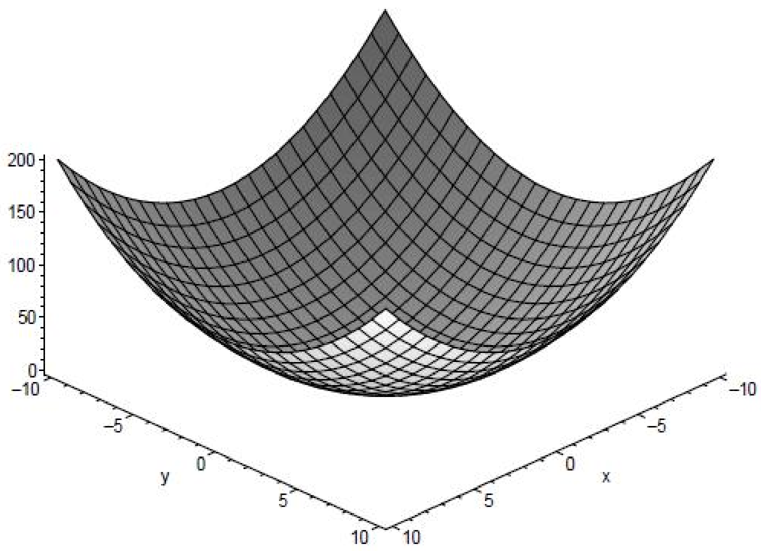
Example 5.4 Let $g : \mathbb{R}^2 \to \mathbb{R}$ be the polynomial function defined by

$$f(x,y) = x^2+y^2$$

The function g has a minimum at $(0,0)$. ∎

Definition 5.1.2 [Graphical Representation] Let f be a function of two variables defined on a domain D. The set of points with coordinates (x,y,z) where $z = f(x,y)$, for (x,y) ranging over D, is called the surface of equation $z = f(x,y)$.
To represent it, we plot the points with coordinates $M(x,y,f(x,y))$.

Example 5.5 Representation of the surface of equation $z = x^2 + y^2$



∎

5.1.2 Limit

Definition 5.1.3 Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ be a function defined in the neighborhood of (a_1,a_2). We say that f tends to $\ell \in \mathbb{R}$ at (a_1,a_2) if

$$\forall \varepsilon > 0, \exists \eta > 0, \forall (x_1,x_2) \in D, \|(x_1,x_2)-(a_1,a_2)\| \leqslant \eta \implies |f(x_1,x_2)-\ell| \leqslant \varepsilon$$

and we denote $\lim\limits_{(x_1,x_2)\to(a_1,a_2)} f(x_1,x_2) = \ell$

R

- The quantity $\|(x_1,x_2)-(a_1,a_2)\|$ denotes the distance between the variable (x_1,x_2) and the point (a_1,a_2). This distance can be measured by several norms such as: $\|(x_1,x_2)-(a_1,a_2)\| = \sqrt{(x_1-a_1)^2+(x_2-a_2)^2}$.
- This definition, as well as those of other cases ($+\infty$ ....), are analogous to the definitions of limits of real functions of a real variable.
- This approach does not consist of making x_1 tend to a_1 and then x_2 tend to a_2, which would be incorrect. Here, it is the pair (x_1,x_2) that tends to (a_1,a_2), so in practice we often use the following theorem:

Theorem 5.1.1 Let f be a function from $\mathbb{R}^2$ to $\mathbb{R}$ and $\ell \in \mathbb{R}$. Let $X=(x_1,x_2)$ and $A=(a_1,a_2)$. If there exists a function g such that $|f(X)-\ell| \leqslant g(X) \underset{X\to A}{\to} = 0$, then:

$$\lim_{X\to A} f(X) = \ell$$

■ **Example 5.6** Let's study $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x^2+y^2}$

$$\left|\frac{x^3+y^3}{x^2+y^2}\right| \leqslant \left|\frac{x^3}{x^2+y^2}\right| + \left|\frac{y^3}{x^2+y^2}\right| \leqslant \left|\frac{x^3}{x^2}\right| + \left|\frac{y^3}{y^2}\right| \leqslant |x|+|y| \underset{(x,y)\to(0,0)}{\longrightarrow} 0$$

■

■ **Example 5.7** Let's study $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}}$

$$\left|\frac{xy}{\sqrt{x^2+y^2}}\right| \leqslant \frac{1}{2}\sqrt{x^2+y^2} \underset{(x,y)\to(0,0)}{\longrightarrow} 0$$

■

5.1.3 Continuity

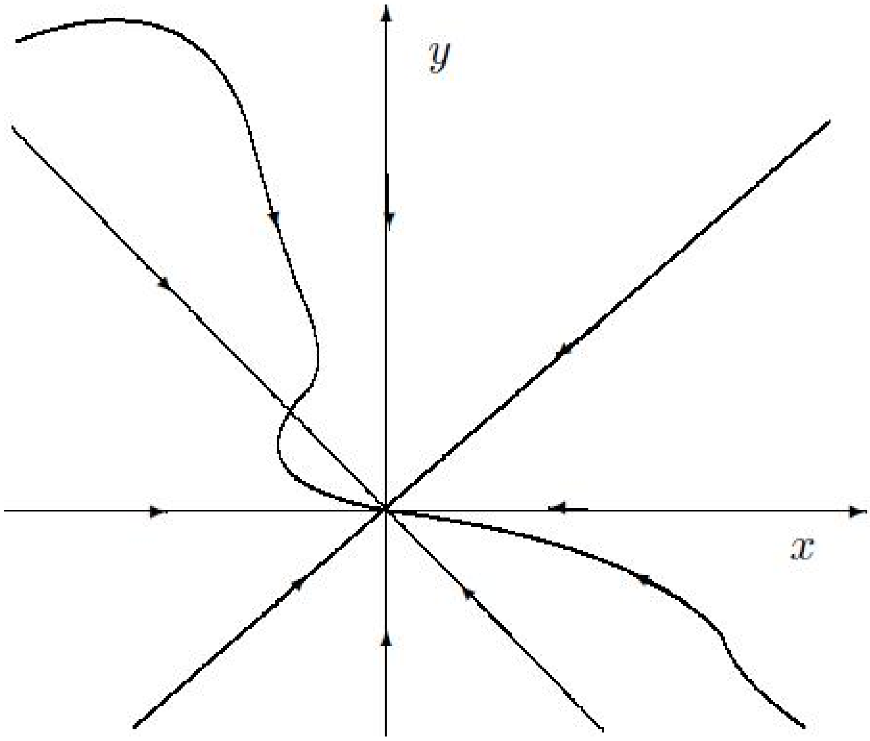
Definition 5.1.4 We say that $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous at a point $a=(a_1,a_2) \in D$ if $f \underset{a}{\to} f(a)$.
We say that f is continuous on D if f is continuous at every point $a \in D$.
We denote $\mathscr{C}(D,\mathbb{R})$ the set of real functions defined and continuous on D.

■ **Example 5.8** Constant functions and polynomial functions are continuous.

The function $(x,y) \to \dfrac{xy+y\sin x}{x^2+y^2+1}$ is continuous on $\mathbb{R}^2$ (by operations on continuous functions).

■

■ **Example 5.9** The case of discontinuity i.e. divergence of the limit If f is a function from $\mathbb{R}^2$ to $\mathbb{R}$ discontinuous at $x_0 \in \mathbb{R}^2$, how will this behavior be characterized?
In the case of a single variable, discontinuity occurs when the left-hand limit is different from the right-hand limit. However, in two dimensions on a plane, there are infinitely many ways to approach a. If $a=(0,0)$ for example : ■



R To demonstrate that a function f of two variables is discontinuous at $a \in \mathbb{R}^2$, it is necessary to find two different directions leading to different limits, or alternatively, to cause the function to diverge in a particular direction.

R A parametric arc is used to represent curves in the plane or space. We use a parameter t. The curve is given by the points $M(t)$ with coordinates $(x(t), y(t))$, where x and y are two functions of t and by an interval I, such that $t \in I$.
In general, $x(t)$ and $y(t)$ will often denote polynomial functions.
The idea is that we should be able to make $(x(t), y(t))$ converge to the point of discontinuity as t tends to a certain value, usually 0.

■ **Example 5.10** Let's study the continuity of the function f defined by

$$f(x,y) = \begin{cases} \dfrac{(x+y)^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1, \quad \text{otherwise} \end{cases}$$

f is continuous on $\mathbb{R}^2 \backslash \{(0,0)\}$ by operations on continuous functions.
However, along the direction $(x(t), y(t)) = (t,t)$ we have

$$f(t,t) \underset{t \to 0}{\to} 2 \neq f(0,0)$$

so f is not continuous at $(0,0)$. ■

■ **Example 5.11** Let's study the continuity extension at $(0,0)$ of the function

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2} \text{ defined on } \mathbb{R}^2 \backslash \{(0,0)\}$$

We have $f(t,0) = 1 \underset{t \to 0}{\to} 1$ and $f(0,t) = -1 \underset{t \to 0}{\to} -1$.
We can assert the non-existence of the studied limit, and consequently f cannot be extended by continuity at $(0,0)$. ■

R If f is a function of two variables and we want to study the possible limit of $f(x,y)$ as $(x,y) \to (x_0,y_0)$, several cases may arise:

- If f is continuous at (x_0,y_0): we prove convergence using an enclosure or other techniques not discussed here, such as the passage to polar coordinates or Taylor expansion.

- If f is discontinuous at (x_0,y_0): it is a matter of finding the right parametric arcs that will prove that in two different directions, f has two different limits.

5.1.4 Partial Derivatives

Definition 5.1.5 Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ and $a = (a_1,a_2) \in D$.
The first partial application of f at point a is the function of a real variable $f(.,a_2) : x \to f(x,a_2)$.
The second partial application of f at point a is the function of a real variable $f(a_1,.) : y \to f(a_1,y)$.

■ **Example 5.12** The first partial application of $f : (x,y) \to x^2 + y^2$ at point $(2,3)$ is the function $f(.,3) : x \to x^2 + 9$. ■

Proposition 5.1.2 If $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous then the partial applications of f at every point in D are continuous.

R The converse is false, the continuity of partial applications does not ensure the continuity of the function.

R The definition of the derivative of a single-variable function can be applied to the partial applications of a function with two (or more) variables.
The partial derivatives of a function of two variables x and y are calculated as follows:

- With respect to x: we consider y as constant and differentiate the function as a function of x.

- With respect to y: we consider x as constant and differentiate with respect to y.

The partial derivative of f with respect to x is still a function of two variables denoted $\dfrac{\partial f}{\partial x}$.
Similarly, the partial derivative of a function f with respect to y is denoted $\dfrac{\partial f}{\partial y}$.
The partial derivatives of f are the derivatives of its partial applications.

■ **Example 5.13** Consider the function defined on $\mathbb{R}^2$ by $f(x,y) = x^2y + x$.
We have $\dfrac{\partial f}{\partial x}(x,y) = 2xy + 1$ and $\dfrac{\partial f}{\partial y}(x,y) = x^2$.
■

■ **Example 5.14** Consider the function defined on $\mathbb{R} \times]0,+\infty[$ by $f(x,y) = x^3 \ln y$.
We have $\dfrac{\partial f}{\partial x}(x,y) = 3x^2 \ln y$ and $\dfrac{\partial f}{\partial y}(x,y) = \dfrac{x^3}{y}$.
■

■ **Example 5.15** Consider the function defined on $\mathbb{R}^2$ by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0, & \text{otherwise} \end{cases}$$

Let's calculate the partial derivatives of f at $(0,0)$.
We have $\quad \dfrac{f(h,0) - f(0,0)}{h} = 0 \underset{h \to 0}{\to} 0.$
We deduce the existence of the first partial derivative at $(0,0)$ and $\dfrac{\partial f}{\partial x}(0,0) = 0$.
Similarly we obtain $\dfrac{\partial f}{\partial y}(0,0) = 0$.
■

R In the previous example, the function is not continuous at $(0,0)$ because

$$f(t,t) \underset{t \to 0}{\to} \frac{1}{2} \neq f(0,0)$$

Unlike real single-variable functions, the existence of partial derivatives does not ensure continuity at the point..

R Let $f : \mathbb{R} \to \mathbb{R}$. We know how to define (provided they exist...) $f', f'', f^{(3)}$. Similarly, it is easy to define higher order derivatives for a function f of two real variables:

- $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right)$: we differentiate twice with respect to x.
- $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)$: we differentiate twice with respect to y.
- $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)$: we differentiate with respect to y, then with respect to x.
- $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$: we differentiate with respect to x, then with respect to y.

■ **Example 5.16** Let g be the function defined on $\mathbb{R}^2$ by: $g(x,y) = x^3y^2$.
Clearly, we have: $\frac{\partial^2 g}{\partial x^2}(x,y) = 6xy^2$, $\frac{\partial^2 g}{\partial y^2}(x,y) = 2x^3$, $\frac{\partial^2 g}{\partial x \partial y}(x,y) = 6x^2y$. ■

Definition 5.1.6 We say that a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is of class $\mathscr{C}^1$ if its partial derivatives exist and are continuous on D. We denote $\mathscr{C}^1(D,\mathbb{R})$ the set of such functions.

Theorem 5.1.3 Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ be a function of class $\mathscr{C}^1$ and $a = (a_1,a_2) \in D$.
For any $h = (h_1,h_2)$ such that $a+h \in D$, we can write:

$$f(a+h) = f(a) + h_1 \frac{\partial f}{\partial x}(a) + h_2 \frac{\partial f}{\partial y}(a) + o(\|h\|)$$

This relation is called the first-order Taylor expansion of the function f at the point a.

Proposition 5.1.4 If f is a function of class $\mathscr{C}^1$ then f is continuous.

Definition 5.1.7 We say that a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is of class $\mathscr{C}^2$ if its second-order partial derivatives exist and are continuous on D. We denote $\mathscr{C}^2(D,\mathbb{R})$ the set of such functions.

R In general $\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$.

However, we have the following result:

Theorem 5.1.5 — Schwarz's Theorem. If f is a function from $\mathbb{R}^2$ to $\mathbb{R}$, of class $\mathscr{C}^2$ on an open set $D \subset \mathbb{R}^2$, at every point of D, we have: $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

■ **Example 5.17** Let's consider f the function defined on $\mathbb{R}^2$ by:

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0, \quad \text{otherwise} \end{cases}$$

We can verify that $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$, hence $f \notin \mathscr{C}^2(\mathbb{R}^2,\mathbb{R})$. ■

DIFFERENTIABILITY

5.2 Differentiability

Definition 5.2.1 We say that $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $a = (a_1,a_2) \in D$ if for any

$h = (h_1, h_2)$, vector in $\mathbb{R}^2$, we have as $\|h\| \to 0$

$$f(a+h) = f(a) + h_1 \frac{\partial f}{\partial x}(a) + h_2 \frac{\partial f}{\partial y}(a) + o(\|h\|)$$

i.e. f admits a first-order Taylor expansion at a.
In this case, the linear map

$$\begin{aligned} d_a f : \quad & \mathbb{R}^2 && \longrightarrow \mathbb{R} \\ & (h_1, h_2) && \longrightarrow h_1 \frac{\partial f}{\partial x}(a) + h_2 \frac{\partial f}{\partial y}(a) \end{aligned}$$

is called the differential of f at a.

R

- If f is differentiable at a then it is continuous at a.
- The existence of partial derivatives of f does not justify its differentiability at a.
- If f is of class $\mathscr{C}^1$ on D then it is differentiable at every point in D.

■ **Example 5.18** Let's consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = xy$.
Since f is a polynomial function, it is of class $\mathscr{C}^\infty$ and hence it is differentiable at every point in $\mathbb{R}^2$.
Let's calculate, for example, $d_{(2,3)}f$, the differential of f at $(2,3)$. For any $(h_1, h_2) \in \mathbb{R}^2$

$$d_{(2,3)}f(h_1, h_2) = h_1 \frac{\partial f}{\partial x}(2,3) + h_2 \frac{\partial f}{\partial y}(2,3) = 3h_1 + 2h_2.$$

Let's call dx and dy the linear forms "projection on the axes" defined as follows: 4.5cm

$$\begin{aligned} dx : \quad & \mathbb{R}^2 && \longrightarrow \mathbb{R} \\ & (h_1, h_2) && \longrightarrow h_1 \end{aligned}$$

4.5cm

$$\begin{aligned} dy : \quad & \mathbb{R}^2 && \longrightarrow \mathbb{R} \\ & (h_1, h_2) && \longrightarrow h_2 \end{aligned}$$

then the linear map $d_{(2,3)}f$ is written as

$$d_{(2,3)}f = 3dx + 2dy$$

■

R Concepts like limit, continuity, partial derivatives, differentiability, Taylor expansion... of a function of two variables can be generalized to functions of several variables.

R In summary, we have:

$$\begin{array}{ccccc} f \in C^1(D) & \Longrightarrow & f \text{ is differentiable on } D & \Longrightarrow & f \in C^0(D) \\ & & \Downarrow & & \\ & & f \text{ has partial derivatives} & & \end{array}$$

DOUBLE AND TRIPLE INTEGRALS

5.3 Double and Triple Integrals

5.3.1 Double Integrals

Theorem 5.3.1 — Fubini's Theorem. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be continuous on the rectangle $\mathscr{R} = [a,b] \times [c,d]$ $(a < b$ and $c < d)$. Then:

$$\int_a^b \int_c^d f(x,y) \cdot dx \cdot dy = \int_c^d \int_a^b f(x,y) \cdot dx \cdot dy.$$

■ **Example 5.19** Calculate $J = \int_0^1 \int_3^5 xy \cdot dx \cdot dy$

$\int_3^5 xy \cdot dx = y \left[\frac{1}{2}x^2\right]_3^5 = 8y$ and $\int_0^1 8y \cdot dy = 4$. Therefore, $J = 4$. ■

■ **Example 5.20** Let U be a bounded region of $\mathbb{R}^2$, and f be a function from $\mathbb{R}^2$ to $\mathbb{R}$, defined and continuous on U.
U is a region included in a rectangle of the form $\mathscr{R} = [a,b] \times [c,d]$.

We want to integrate f over U.
Let's define:

$$\tilde{f}(x,y) = \begin{cases} f(x,y) \text{ if } (x,y) \in U \\ 0 \qquad \text{ if } (x,y) \in \mathscr{R} \backslash U. \end{cases}$$

Then:

$$\iint_U f(x,y) \cdot dx \cdot dy = \iint_{\mathscr{R}} \tilde{f}(x,y) \cdot dx \cdot dy.$$

Often, we will deal with domains of different shapes:

$$\begin{cases} a \le x \le b, \\ \varphi_1(x) \le y \le \varphi_2(x). \end{cases}$$

Then, we will write:

$$\iint_U f(x,y) \cdot dx \cdot dy = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y) \cdot dy \cdot dx.$$

If the domain is of the form:

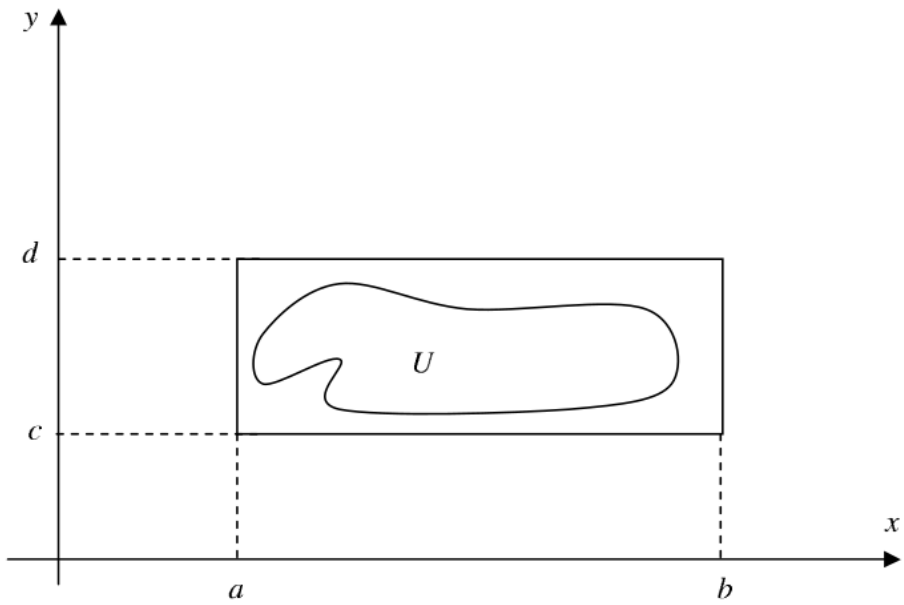
$$\begin{cases} a \le y \le b, \\ \varphi_1(y) \le x \le \varphi_2(y). \end{cases}$$

Then, we will write:

$$\iint_U f(x,y) \cdot dx \cdot dy = \int_a^b \int_{\varphi_1(y)}^{\varphi_2(y)} f(x,y) \cdot dx \cdot dy.$$

■

R The following figure illustrates the general calculation principle, when the domain is bounded.



■ **Example 5.21** Compute $I = \int_3^4 \int_1^2 \frac{dx \cdot dy}{(x+y)^2}$

Prior to computation, notice that this integral is not improper since $(x,y) \longmapsto \frac{1}{(x+y)^2}$ is continuous over the rectangle $[3,4] \times [1,2]$. Thus, we can write:

$$\begin{aligned} \mathscr{A} &= \int_3^4 \left[\frac{-1}{x+y} \right]_{y=1}^{y=2} dx = \int_3^4 \left(\frac{1}{x+1} - \frac{1}{x+2} \right) dx \\ &= \left[\ln \left(\frac{x+1}{x+2} \right) \right]_3^4 = \ln \frac{25}{24} \end{aligned}$$

■

R Suppose we want to integrate over the rectangle $\mathscr{R} = [a,b] \times [c,d]$, where $a < b$ and $c < d$, the continuous function f, of two variables x and y, defined by a relation of the form:

$f(x,y) := g(x)h(y)$, where g and h are two continuous functions of one variable. Then, we can write:

$$\int_a^b \int_c^d f(x,y)\cdot dx\cdot dy = \int_a^b \int_c^d g(x)h(y)\cdot dx\cdot dy = \left(\int_a^b g(x)\cdot dx\right)\left(\int_c^d h(y)\cdot dy\right).$$

Proposition 5.3.2 For f a continuous function over a domain $U \subset \mathbb{R}^2$ and real-valued. Let $\mathscr{D}_1$ and $\mathscr{D}_2$ be two disjoint domains ($\mathscr{D}_1 \cap \mathscr{D}_2 = \emptyset$) included in U. Then:

$$\iint_{\mathscr{D}_1\cup\mathscr{D}_2} f = \iint_{\mathscr{D}_1} f + \iint_{\mathscr{D}_2} f.$$

Similarly, we can write formally:

$$\iint_{\mathscr{D}} (f+g) = \iint_{\mathscr{D}} f + \iint_{\mathscr{D}} g.$$

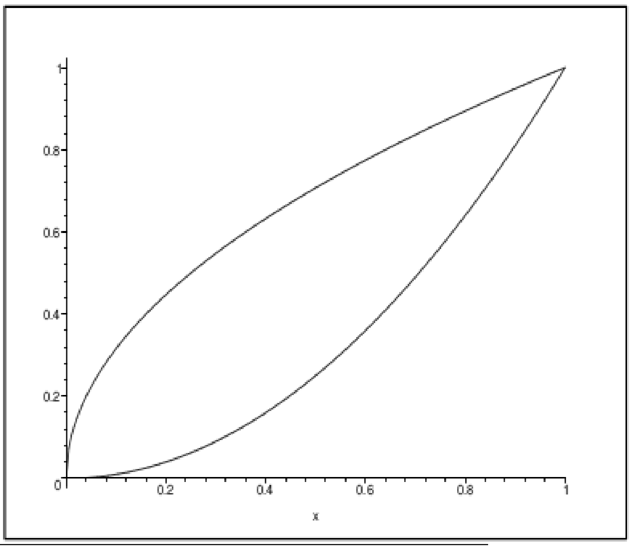
R Double integrals also allow us to calculate areas. The principle is the same as in dimension 1. If $\mathscr{D}$ is a bounded domain of $\mathbb{R}^2$, and if we denote $\mathscr{A}(\mathscr{D})$, its area, then we have:

$$\mathscr{A}(\mathscr{D}) = \iint_{\mathscr{D}} dx\cdot dy.$$

■ **Example 5.22** We call $\mathscr{D}$ the set of points (x,y) such that:

$$\begin{cases} 0 \leq x \leq 1 \\ x^2 \leq y \leq \sqrt{x} \end{cases}$$

We want to calculate the area of the domain $\mathscr{D}$, which corresponds to the lune depicted above.



We denote $\mathscr{A}$ this area. We have:

$$\mathscr{A} = \iint_{\mathscr{D}} dx\cdot dy = \int_0^1 \int_{x^2}^{\sqrt{x}} dy\cdot dx = \int_0^1 \left(\sqrt{x}-x^2\right) dx$$
$$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right]_0^1 = \frac{1}{3}$$

■

Theorem 5.3.3 Let f be a function continuous on a bounded domain U of $\mathbb{R}^2$.
Sometimes, we can use a change of variables to compute
$\iint_U f(x,y)\cdot dx\cdot dy$.
Let φ be a $\mathscr{C}^1$-diffeomorphism on U, that is:

- $\varphi : (x,y) \longmapsto (u,v)$ bijective of class $\mathscr{C}^1$ on U.
- φ^{-1}, the inverse bijection, is also of class $\mathscr{C}^1$ on $U' = \varphi(U)$.

Then we have:

$$\iint_U f(x,y)\cdot dx\cdot dy = \iint_{U'} g(u,v)|\det J|\cdot du\cdot dv,$$

where J denotes the Jacobian matrix of φ^{-1}, that is:

$$\det J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}.$$

■ **Example 5.23** Polar Coordinates: If $x = \rho\cos\theta$ and $y = \rho\sin\theta$, $\det J = \begin{vmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{vmatrix} = \rho$.
Using the same notations as in the theorem above, we have:

$$\iint_U f(x,y)\cdot dx\cdot dy = \iint_{U'} g(\rho,\theta)\rho d\rho d\theta.$$

■

■ **Example 5.24** Calculating $I = \iint_U \dfrac{xy}{\sqrt{x^2+y^2}}\cdot dx\cdot dy$, where $U = \left\{(x,y)\in\mathbb{R}^2 : x\geq 0, y\geq 0, a^2\leq x^2+y^2\leq b^2\right\}$

A switch to polar coordinates immediately yields: $I = \iint_{U'} \rho^2\cos\theta\sin\theta d\rho d\theta$, where $U' = [a,b]\times\left[0,\dfrac{\pi}{2}\right]$. (Draw a picture) Then:

$$I = \int_a^b \rho^2 d\rho \int_0^{\frac{\pi}{2}} \cos\theta\sin\theta d\theta = \frac{b^3-a^3}{6}.$$

■

5.3.2 Triple Integrals

The definition we provided extends without additional difficulty to triple integrals. We will illustrate the different methods by computing in two ways the volume of a sphere. It is actually the volume of the ball bounded by the sphere. Let $\mathscr{B}$ be the closed ball in $\mathbb{R}^3$ with center O and radius R.

$$\mathscr{B} = \left\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2\leq R^2\right\}.$$

The volume of $\mathscr{B}$ is: $\mathscr{V} = \iiint_{\mathscr{B}} dx \cdot dy \cdot dz$.

- **Stacked Integration** (Method 1):
  We reduce it to a double integral:

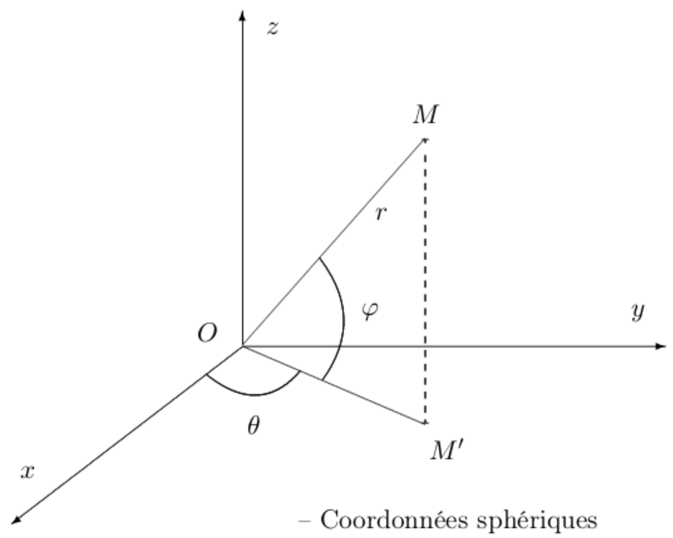
$$\mathscr{V} = \iint_{\mathscr{D}} \left(\int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} dz \right) \cdot dx \cdot dy = 2 \iint_{\mathscr{D}} \sqrt{R^2 - x^2 - y^2} \cdot dx \cdot dy,$$

  where $\mathscr{D}$ denotes the disk in $\mathbb{R}^2$ centered at O with radius R. Then we switch to polar coordinates:

$$\mathscr{V} = 2 \iint_{\mathscr{D}'} \rho \sqrt{R^2 - \rho^2} d\rho d\theta = -\frac{4\pi}{3} \left[\left(R^2 - \rho^2\right)^{\frac{3}{2}} \right]_0^R = \frac{4\pi R^3}{3}$$

Calculating the triple integral by a change of variables, here are the spherical coordinates:

$$\begin{cases} x = \rho \cos\varphi \cos\theta \\ y = \rho \cos\varphi \sin\theta. \\ z = \rho \sin\varphi \end{cases}$$



– Coordonnées sphériques

We choose ρ, φ and θ such that : $\rho \in [0, r]; \theta \in [0, 2\pi]; \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. This choice is easily justified from the figure. The Jacobian matrix of this transformation is:

$$J = \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \cos\varphi\cos\theta & -\rho\sin\theta\cos\varphi & -\rho\sin\varphi\cos\theta \\ \cos\varphi\sin\theta & \rho\cos\theta\cos\varphi & -\rho\sin\varphi\sin\theta \\ \sin\varphi & 0 & \rho\cos\varphi \end{pmatrix}.$$

The determinant of J is: $\det J = \rho^2 \cos\varphi$ (Sarrus' rule). Hence:

$$\mathscr{V} = \iiint_{\mathscr{B}'} \rho^2 \cos\varphi d\rho d\varphi d\theta = \int_0^R \rho^2 d\rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi d\varphi \int_0^{2\pi} d\theta.$$

This immediately leads to $\mathscr{V} = \dfrac{4\pi R^3}{3}$.

More generally, let's state the theorem of change of coordinates in dimension 3, of which the best application example is the calculation of the volume of the sphere of center O and radius R performed above:

Theorem 5.3.4 — Change of Variables. Let $\psi : U \subset \mathbb{R}^3 \longrightarrow V \subset \mathbb{R}^3$, a $\mathscr{C}^1$-diffeomorphism such that :

$$\psi(u,v,w) = (P(u,v,w), Q(u,v,w), R(u,v,w)).$$

If $\mathscr{D} \subset U$, we denote $\mathscr{D}' = \psi(\mathscr{D})$. We can still write $\mathscr{D} = \psi^{-1}(\mathscr{D}')$, and if f is a continuous function from U to $\mathbb{R}$, then:

$$\iiint_{\mathscr{D}'} f(x,y,z) \cdot dx \cdot dy \cdot dz = \iiint_{\mathscr{D}} f(\psi(u,v,w)) \left| J_\psi(u,v,w) \right| \cdot du \cdot dv \cdot dw,$$

where J_ψ denotes the determinant:

$$J_\psi := \det \begin{pmatrix} \dfrac{\partial P}{\partial u} & \dfrac{\partial P}{\partial v} & \dfrac{\partial P}{\partial w} \\ \dfrac{\partial P}{\partial u} & \dfrac{\partial Q}{\partial v} & \dfrac{\partial Q}{\partial w} \\ \dfrac{\partial P}{\partial u} & \dfrac{\partial R}{\partial v} & \dfrac{\partial R}{\partial w} \end{pmatrix}$$

■ **Example 5.25** In Mechanics, for example, one may need to find the center of gravity of a homogeneous half-sphere:

$$\mathscr{B} := \left\{ (x,y,z) \in \mathbb{R}^3 : z \geq 0,\ x^2 + y^2 + z^2 \leq R^2 \right\}.$$

Let m be the mass of the half-sphere. Let μ be the mass density of the sphere. If O is the center of the coordinate system, the center of gravity G of the solid is defined by the relation:

$$m\overrightarrow{OG} = \iiint_{\mathscr{B}} \mu \overrightarrow{OM} \cdot dx \cdot dy \cdot dz,$$

where M denotes a point in $\mathscr{B}$ with coordinates (x,y,z).

Using symmetry properties of $\mathscr{B}$, it can be easily shown that $x_G = y_G = 0$. We just need to calculate z_G. We have:

$$mz_g = \mu \iiint_{\mathscr{B}} z \cdot dx \cdot dy \cdot dz.$$

We have:

$$mz_g = \mu \iiint_{\mathscr{B}} z \cdot dx \cdot dy \cdot dz.$$

A change to spherical coordinates provides:

$$mz_g = \mu \iiint_{(\rho,\theta,\varphi)\in\Omega} (\rho \sin\varphi) \left(\rho^2 \cos\varphi d\rho d\theta d\varphi\right).$$

with $\Omega = [0,R] \times [0,2\pi] \times \left[0, \frac{\pi}{2}\right]$. The calculation is immediate since the integrand is separable:

$$z_G = \frac{\mu}{m} \left(\int_0^R \rho^3 \cdot d\rho \right) \left(\int_0^{2\pi} d\theta \right) \left(\int_0^{\frac{\pi}{2}} \sin\varphi \cos\varphi \cdot d\varphi \right).$$

Finally, considering that $m = \frac{2}{3}\mu\pi R^3$ (calculation of a mass density), we find: $z_G = \frac{3}{8}R$. ■

6. Solved problems

6.1 Problems

Problem 6.1 Consider the matrix

$$M = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

1. Calculate $M^3 - 2M^2 + 2M$.
2. Deduce from the above that the matrix M is invertible; then provide M^{-1}.
3. Retrieve M^{-1} using the adjugate matrix (Comatrix).

Problem 6.2 Consider the matrix: $A = \begin{pmatrix} m & 1 & m+1 \\ 0 & 1 & 2 \\ m & 0 & -1 \end{pmatrix}, \quad m \in \mathbb{R}$.

1. For which values of m is the matrix A invertible?
2. In the case where $m = 2$, calculate the inverse of A.

Problem 6.3 Solve and discuss, depending on the parameter m, the system of equations:

$$(S_1):\left\{\begin{array}{rcl} -mx+y-mz & = & 0 \\ x+my-z & = & -m \\ 2x+y-z & = & 1 \end{array}\right. ,$$

Problem 6.4

1. Trouver a,b et c de $\mathbb{R}$ tel que : $\dfrac{1}{(1+x)(1+x^2)}=\dfrac{a}{1+x}+\dfrac{bx+c}{(1+x^2)}$
2. Calculer l'intégrale suivante : $I=\displaystyle\int\frac{dx}{(1+x)(1+x^2)}$.
3. En déduire l'intégrale suivante : $J=\displaystyle\int\frac{\arctan x}{(1+x)^2}dx$.
4. Résoudre l'équation différentielle suivante en précisant son type :

$$(x+1)y'+y=\frac{\arctan x}{(1+x)^2}........(E)$$

   En déduire la solution particulière de (E).
5. Résoudre l'équation différentielle suivante en précisant son type :

$$y''+y'-y=0.$$

6.2 Solutions

Solution of problem 5.1

1.

$$M^2=MM=\begin{pmatrix}0&1&-1\\0&1&1\\1&0&1\end{pmatrix}\begin{pmatrix}0&1&-1\\0&1&1\\1&0&1\end{pmatrix}=\begin{pmatrix}-1&1&0\\1&1&2\\1&1&0\end{pmatrix}$$

$$M^3=M^2M=\begin{pmatrix}-1&1&0\\1&1&2\\1&1&0\end{pmatrix}\begin{pmatrix}0&1&-1\\0&1&1\\1&0&1\end{pmatrix}=\begin{pmatrix}0&0&2\\2&2&2\\0&2&0\end{pmatrix}$$

So

$$M^3-2M^2+2M=\begin{pmatrix}0&0&2\\2&2&2\\0&2&0\end{pmatrix}-2\begin{pmatrix}-1&1&0\\1&1&2\\1&1&0\end{pmatrix}+2\begin{pmatrix}0&1&-1\\0&1&1\\1&0&1\end{pmatrix}$$

$$=\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}$$

2. We have already established that $M^3-2M^2+2M=2I$, so $\frac{1}{2}M\left(M^2-2M+2I\right)=I$, and since $MM^{-1}=M^{-1}M=I$, we deduce

$$M^{-1}=\frac{1}{2}\left(M^2-2M+2I\right)=\frac{1}{2}\begin{pmatrix}1&-1&2\\1&1&0\\-1&1&0\end{pmatrix}$$

3. M^{-1} by using the comatrix :
We have $|M|=2$.

$$M^{-1}=\frac{1}{|M|}\operatorname{com}(M)=\frac{1}{2}\begin{pmatrix}1&1&-1\\-1&1&1\\2&0&0\end{pmatrix}^t=\frac{1}{2}\begin{pmatrix}1&-1&2\\1&1&0\\-1&1&0\end{pmatrix}$$

Solution of problem 5.2

1. We have : $|A|=-m^2$ and then A is invertible iff $m\neq 0$.
2. After computation :

$$A^{-1}=\begin{pmatrix}1/4&-1/4&1/4\\-1&2&1\\1/2&-1/2&-1/2\end{pmatrix}.$$

Solution of problem 5.3

$$(S_1)\Leftrightarrow AX=B \quad \text{with } A=\begin{pmatrix}-m&1&-m\\1&m&-1\\2&1&-1\end{pmatrix};B=\begin{pmatrix}0\\-m\\1\end{pmatrix}$$

$$\det(A)=\begin{vmatrix}-m&1&-m\\1&m&-1\\2&1&-1\end{vmatrix}=\begin{vmatrix}-m&1&0\\1&m&-2\\2&1&-3\end{vmatrix}=-m\begin{vmatrix}m&-2\\1&-3\end{vmatrix}-\begin{vmatrix}1&-2\\2&-3\end{vmatrix}=-m(-3m+2)-1.$$

So (S_1) admits a unique solution $\Leftrightarrow m\in\mathbb{R}\backslash\left\{-\frac{1}{3},1\right\}$.
If $m\in\left\{-\frac{1}{3},1\right\}$, the system is incompatible since

- for $m=1$:

$$\left.\begin{array}{r}\text{from } (L_2)-(L_1),\ \text{on a } 2x=1\\ \text{from } (L_3)-(L_2),\ \ \text{we have } x=2\end{array}\right\}\Longrightarrow 2=1/2.$$

- for $m=-\dfrac{1}{3}$:

$$\left.\begin{array}{r}\text{From } (L_2)+(L_3),\ \text{ we have } 10x-8z=4\\ \text{From } (L_1)+3(L_2),\ \text{ we have } 10x-8z=3\end{array}\right\}\implies 4=3.$$

For $m\in\mathbb{R}\backslash\left\{-\dfrac{1}{3},1\right\}$, we solve using Cramer's method:

$$x=\frac{\begin{vmatrix}0&1&-m\\-m&m&-1\\1&1&-1\end{vmatrix}}{3\left(m+\frac{1}{3}\right)(m-1)}=\frac{\begin{vmatrix}0&1&-m\\0&2m&-1-m\\1&1&-1\end{vmatrix}}{3\left(m+\frac{1}{3}\right)(m-1)}=\frac{-m-1+2m^2}{3\left(m+\frac{1}{3}\right)(m-1)},$$

$$y=\frac{\begin{vmatrix}-m&0&-m\\1&-m&-1\\2&1&-1\end{vmatrix}}{\det A}=\frac{\begin{vmatrix}0&0&-m\\2&-m&-1\\3&1&-1\end{vmatrix}}{\det A}=\frac{-2m-3m^2}{3\left(m+\frac{1}{3}\right)(m-1)},$$

$$z=\frac{\begin{vmatrix}-m&1&0\\1&m&-m\\2&1&1\end{vmatrix}}{\det A}=\frac{\begin{vmatrix}0&1&0\\1+m^2&m&-m\\2+m&1&1\end{vmatrix}}{\det A}=\frac{-\left(m^2+1\right)-2m-m^2}{\det A}=\frac{-2m^2-2m-1}{(3m+1)(m-1)}.$$

$$S=\left\{\left(\frac{2m^2-m-1}{(3m+1)(m-1)};\frac{-3m^2-2m}{(3m+1)(m-1)};\frac{-2m^2-2m-1}{(3m+1)(m-1)}\right\}\right\}.$$

Solution of problem 5.4

1. $a=c=1/2,\ \ b=-1/2$
2. Calculating :

$$I=\int\frac{dx}{(1+x)\left(1+x^2\right)}$$

We decompose to simple elements :

$$I=\int\frac{dx}{(1+x)\left(1+x^2\right)}=\int\left(\frac{A}{1+x}+\frac{Bx+C}{1+x^2}\right)dx=\int\left(\frac{\frac{1}{2}}{1+x}+\frac{-\frac{1}{2}x+\frac{1}{2}}{1+x^2}\right)dx$$
$$=\frac{1}{2}\ln|1+x|-\frac{1}{4}\ln\left(1+x^2\right)+\frac{1}{2}\arctan x+C, C\in\mathbb{R}$$

3. Deducing :

$$J=\int\frac{\arctan x}{(1+x)^2}dx$$

We integrate by parts to get

$$J=\int\frac{\arctan x}{(1+x)^2}dx=-\frac{\arctan x}{1+x}+\int\frac{dx}{(1+x)\left(1+x^2\right)}=-\frac{\arctan x}{1+x}+I$$

4. (E) is a linear differential equation of order 1.
   The general solution of (E) : $y_G=y_0+y_P$,
   y_0 is the solution of $(x+1)y'+y=0$.
   - If $y=0$: is a solution of (E_0).
   - Si $y\neq 0, \int\frac{dy}{y}=-\int\frac{dx}{1+x}\Longrightarrow y=\frac{K}{1+x}, K\in\mathbb{R}^*$, then $y_0=\frac{K}{1+x}, K\in\mathbb{R}$.

   We use the method of vaiation of constant, let :

$$y_G=\frac{K(x)}{1+x}\Longrightarrow y_G'=\frac{K'(x)(1+x)-K(x)}{(1+x)^2}.$$

Replace in (E) to get :

$$K'(x)=\frac{\arctan x}{(1+x)^2}\Longrightarrow K(x)=\int\frac{\arctan x}{(1+x)^2}dx=J.$$

Therefore :

$$K(x)=-\frac{\arctan x}{1+x}+\frac{1}{2}\ln|1+x|-\frac{1}{4}\ln\left(1+x^2\right)+\frac{1}{2}\arctan x+C, C\in\mathbb{R},$$

So :

$$y_G=-\frac{\arctan x}{(1+x)^2}+\frac{1}{2}\frac{\ln|1+x|}{1+x}-\frac{1}{4}\frac{\ln\left(1+x^2\right)}{1+x}+\frac{1}{2}\frac{\arctan x}{1+x}+\frac{C}{1+x}, C\in\mathbb{R}.$$

A particular solution of (E) :

$$y_p=-\frac{\arctan x}{(1+x)^2}+\frac{1}{2}\frac{\ln|1+x|}{1+x}-\frac{1}{4}\frac{\ln\left(1+x^2\right)}{1+x}+\frac{1}{2}\frac{\arctan x}{1+x}.$$

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