Mathematics 01

Dr. Anes MOULAI-KHATIR

People’s Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

University of Oran 2 Mohamed BEN AHMED

Institute of Maintenance and Industrial Safety

Mathematics 01

Author :

Anes MOULAI-KHATIR

"Intended for LMD Bachelor students: (first year Common Core)
Subject: Mathematics"

2023 / 2024

Distribution of Mathematics 1 Program

Program	Mathematics 1
Teaching Units	Fundamental
Subject Teacher	Anes MOULAI-KHATIR
Contact	moulaikhatir.anes@univ-oran2.dz
Total Class Hours	67 hours
Assessment Method	40% Continuous Assessment - 60% Final Exam

Course Canvas

Chapter Title	Contents	Weeks
Mathematical Reasoning	Logical Concepts Direct reasoning Reasoning by contraposition Reasoning by contradiction Reasoning by counter-example Reasoning by induction	01
Sets, Relations, and Functions	Set theory Order relation, Equivalence relations Injective, surjective, bijective applications	02
Real functions of a real variable	Limit, continuity of a function Derivative and differentiability of a function	03
Elementary functions	Power functions Log-Expo functions Trifonometric functions Hyperbolic functions Inverses	03
Series Expansion	Taylor's formula Series Expansion Applications	02
Linear Algebra	Laws and internal composition Vector space, basis, dimension Linear application, kernel, image, rank	04

SITES.GOOGLE.COM/VIEW/MATHIMSI

In the vast expanse of scientific education, a foundational understanding of mathematics is crucial for first-year university students. This course notes document is a carefully curated guide, aimed at engaging and enlightening students in their exploration of mathematical concepts.

The journey begins with an exploration of Mathematical Reasoning (chapter 1), a fundamental intellectual bridge that connects tangible realities to abstract principles. This chapter lays the groundwork for critical thinking, logical deduction, and problem-solving – skills essential for success in scientific fields.

Moving forward, in chapter 2, the narrative unfolds into Sets, Relations, and Applications, revealing the intricate world of mathematical structures and their practical applications across various scientific disciplines. It's here that students learn to navigate complex mathematical relationships and comprehend their relevance in real-world scenarios.

Chapter 3 delves into Functions of a Real Variable, offering a comprehensive study of functions and their applications. From basic understanding to the analysis of continuous and differentiable functions, students gain the tools needed to dissect and manipulate real-world phenomena mathematically.

Elementary Functions are introduced in the subsequent chapter, exploring the foundational blocks of mathematical modeling. From polynomials to exponentials, logarithms, and trigonometric functions, this chapter simplifies complex phenomena, empowering students to express and comprehend intricate concepts with ease.

Series Expansion takes center stage in the fifth chapter, guiding students through the intricate world of approximations. From Taylor series to power series, this section equips students with the ability to systematically and precisely analyze functions, a vital skill for scientific analysis.

The final chapter, Linear Algebra, serves as the culmination of this mathematical journey. Exploring vectors, matrices, eigenvalues, and eigenvectors, students master the language of linear transformations. This chapter lays the groundwork for applications in physics, engineering, and computer science, showcasing the broader relevance of mathematical understanding.

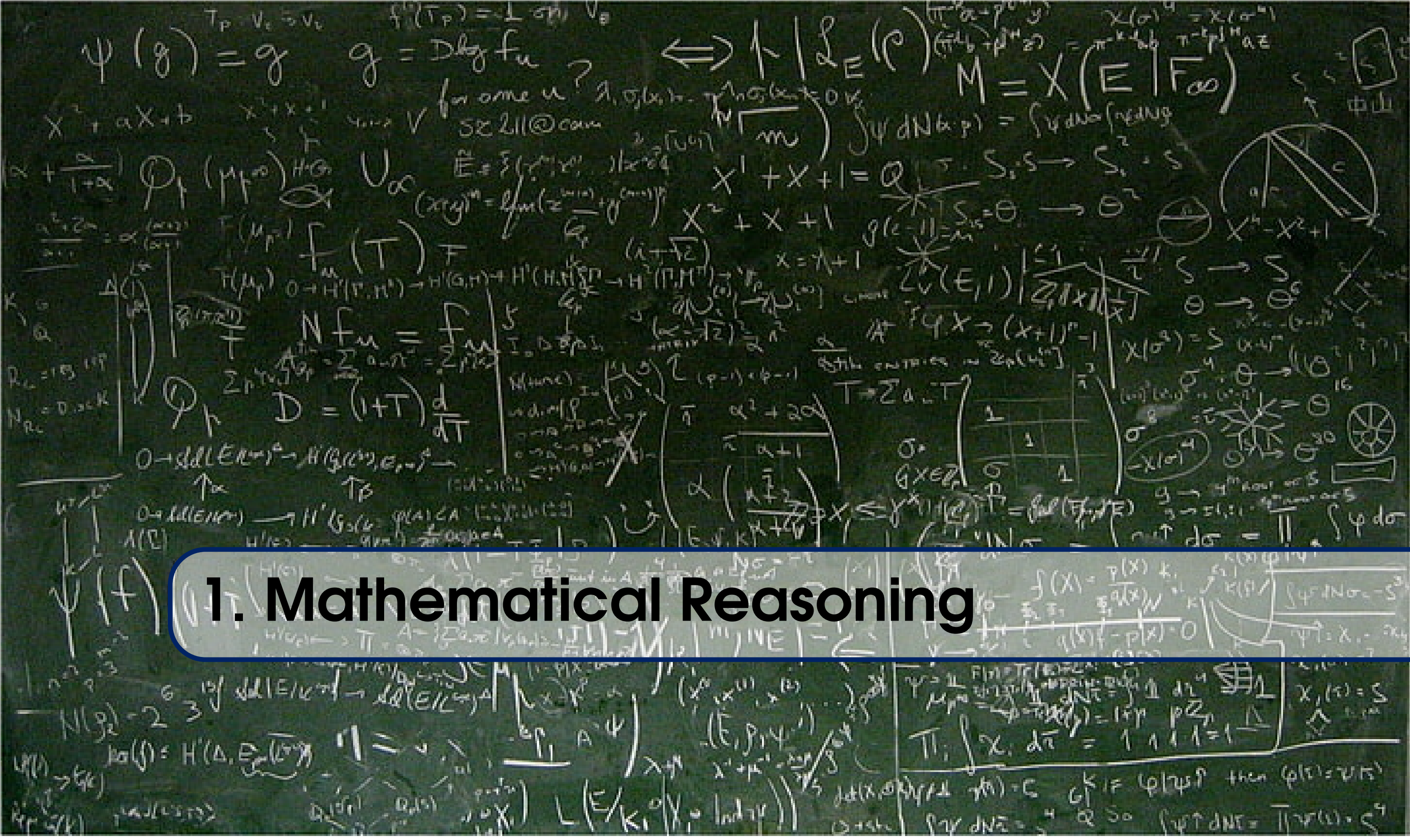
Throughout this document, the narrative is designed to captivate the reader's focus. Examples and solved applications are seamlessly integrated into the text, creating an immersive learning experience. Visual aids and illustrative examples are strategically placed to reinforce comprehension and spark curiosity.

In concluding this exploration, gratitude is extended to all supporters of this work. Educators, with their passion for teaching and commitment to student success, inspire the creation of resources like these. Students, whose thirst for knowledge fuels the pursuit of excellence, play a pivotal role in the success of this endeavor. Special Thanks goes to **Dr. BOUIZEM Youcef** who has done serious contribution to this work

Second version, June 2024

Contents

Course



1. Mathematical Reasoning

In this first chapter, we explore the methodologies and principles that underpin mathematical reasoning, providing you with a solid foundation for approaching and solving mathematical problems with clarity and precision.

1.1 Introduction

This chapter is quite abstract at first reading, but it forms the basis of mathematics along with set theory (chapter 2).
Contemporary mathematics are built in the following way:

- We start with a small number of statements, called axioms, assumed to be true a priori (and therefore not sought to be proven).
- Next, we define the concept of a proof (deciding, for example, what constitutes an implication, an equivalence...).
- Finally, we consider any statement obtained at the end of a proof as "true," and we call such a statement a "theorem."
- Starting from the axioms, we gradually obtain theorems that enrich mathematical theory. Due to the undemonstrated foundations (the axioms), the notion of "truth" in mathematics is a subject of debate.

1.2 Logical concepts

In this section, we unravel the fundamental elements of logic, equipping you with the tools to navigate and comprehend the logical structures that permeate various branches of mathematics.

1.2.1 Assertions

Definition 1.2.1 — Assertion. We call any meaningful mathematical statement that can be true (T) or false (F) an assertion.

Example 1.1 $P =$ « $3 > 2$ » is an assertion that is **true**.
$Q =$ « $2+2=5$ » is an assertion that is **false** ■

(R) When the truth value of an assertion P depends on the values taken by a parameter x (respectively, by several parameters $x, y, \ldots$), it is often denoted as $P(x)$ (respectively, $P(x,y,\ldots)$) to emphasize this dependence.
Sometimes, we refer to this as a **predicate** rather than an **assertion**.

Example 1.2 $P(x) =$ "$x > 0$" is an assertion that depends on a real parameter x. It is true for $x = 2$ and false for $x = -3$. ■

Definition 1.2.2 — Equivalent Assertions. Two assertions P and Q, having the same truth values, are said to be equivalent, denoted as $P \Leftrightarrow Q$.

Example 1.3 "$x > 0$" $\Leftrightarrow$ "$-x < 0$". ■

1.2.2 Negation of an Assertion

Definition 1.2.3 — Negation of an Assertion. The negation of P, denoted as $non(P)$ or $\overline{P}$, is defined to be true when P is false and vice versa. We can also say that the assertion $\overline{P}$ is defined by the truth table:

P	$\overline{\mathbf{P}}$
T	F
F	T

Example 1.4 For $P(x) =$ "$x \geqslant 0$", we have $\overline{P} =$ "$x < 0$". ■

Theorem 1.2.1 For any assertion P, $\overline{(\overline{P})} \Leftrightarrow P$.

Proof. It is clear that $\overline{(\overline{P})}$ and P have the same truth values. ■

1.2.3 Conjunction and Disjunction

Definition 1.2.4 — Conjunction and Disjunction. Let P and Q be two assertions. We can define the assertion "P or Q", denoted as $(P \vee Q)$, and the assertion "P and Q", denoted as $(P \wedge Q)$, using the truth tables below.

P	Q	P∨Q
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	P∧Q
T	T	T
T	F	F
F	T	F
F	F	F

Example 1.5 "$0 \leqslant x \leqslant 1$" $\Leftrightarrow$ "$x \geqslant 0$ and $x \leqslant 1$". "$x \geqslant 0$ or $x \leqslant 0$" is a true assertion for any real x. ■

R The word «or » has two meanings. There is the «exclusive or » which means «either one or the other, but not both » and the «inclusive or » which means «either one or the other, or both ». $\vee$ represents the «inclusive or ».

Theorem 1.2.2 — De Morgan's Law. Let P and Q two assertions.
$(\overline{P \wedge Q}) \Leftrightarrow (\overline{P} \vee \overline{Q})$
$(\overline{P \vee Q}) \Leftrightarrow (\overline{P} \wedge \overline{Q})$

Proof. from the tables of truth. ■

Theorem 1.2.3 Let P, Q and R three assertions.
- $(P \wedge Q) \Leftrightarrow (Q \wedge P)$, $(P \vee Q) \Leftrightarrow (Q \vee P)$.
- $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$, $(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$.
- $(P \wedge Q) \vee R \Leftrightarrow (P \vee R) \wedge (Q \vee R)$, $(P \vee Q) \wedge R \Leftrightarrow (P \wedge R) \vee (Q \wedge R)$.

We say that the «or» and the «and » are commutative, associative and distributive on each other.

1.2.4 Implication

Definition 1.2.5 — Implication. Let P and Q two assertions.
We define the assertion $(P \Rightarrow Q)$ as being true if Q cannot be false when P is true.
In english the implication can be translated to the following: «if... then », «thus »etc.
Precisely, the truth value of the assertion $(P \Rightarrow Q)$ is given by:

P	Q	P $\Longrightarrow$ Q
T	T	T
T	F	F
F	T	T
F	F	T

Example 1.6 For x real number : « $x \geqslant 1 \Rightarrow x^2 \geqslant 1$ » is a true implication.
Otherwise, the implication : « $x^2 \geqslant 1 \Rightarrow x \geqslant 1$ » is false or wrong. ■

R When $P \Rightarrow Q$ is true, we cannot presume or assume the truth value of P.
When $P \Rightarrow Q$ is true :

- if P is true then Q is also true,
- if P is false then we know nothing about the truth value of Q.

To understand «(False $\Rightarrow$ True) is true », let us take the following example :
$2=3$ and $2=1 \Rightarrow 2+2=3+1 \Rightarrow 4=4$.
The starting affirmation is false and we deduce (randomely but by correct reasoning) an affirmation that is true.

A practical consequence of this study is that, if your starting hypothesis is false, even though your reasoning is right, you have no idea about the veracity and the falseness of the conclusions you have reached at the end of your reasoning.

Definition 1.2.6 — Reciprocal implication and Contraposition. When $P \Rightarrow Q$ is true, we say that:
P is a sufficient condition (SC) for Q;
Q is a necessary condition (NC) for P.

Definition 1.2.7
$Q \Rightarrow P$ is known as **Reciprocal Implication** of $P \Rightarrow Q$.
$\overline{Q} \Rightarrow \overline{P}$ is known as **Contraposition** of the implication $P \Rightarrow Q$.

■ **Example 1.7** The contraposition of $(x \geqslant 1) \Rightarrow (x^2 \geqslant 1)$ is $(x^2 < 1) \Rightarrow (x < 1)$. ■

Proposition 1.2.4 $P \Rightarrow Q \Leftrightarrow \overline{P} \vee Q$.
$P \Rightarrow Q \Leftrightarrow \overline{Q} \Rightarrow \overline{P}$. (Every Implication is equivalent to its Contraposition).

Proposition 1.2.5 $\overline{P \Rightarrow Q} \Leftrightarrow P \wedge \overline{Q}$. (By moving to the negation, the symbol of implication disappears).

■ **Example 1.8** The negation of the implication $x \geqslant 1 \Rightarrow x^2 \geqslant 1$ is : « $x \geqslant 1$ and $x^2 < 1$ ». ■

1.2.5 Equivalence

Definition 1.2.8 — Equivalence. Let P and Q two assertions.
We write $P \Leftrightarrow Q$ the assertions $P \Rightarrow Q$ and $Q \Rightarrow P$.
When $P \Leftrightarrow Q$ is true, we write $P \Leftrightarrow Q$, and we say that P is a necessary and sufficient condition (SNC) for Q.
In english, the Equivalence is translated by the expressions «if, and only if»,...
The table of truth for $P \Leftrightarrow Q$ is given by:

P	**Q**	$\mathbf{P \Leftrightarrow Q}$	$\mathbf{P \implies Q}$	$\mathbf{Q \implies P}$	$\mathbf{(P \implies Q) \wedge (Q \implies P)}$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

■ **Example 1.9** for x, y reals :$x^2 = y^2 \Leftrightarrow x = y$ or $x = -y$. ■

Proposition 1.2.6 $(P \Leftrightarrow Q) \Leftrightarrow (\overline{P} \Leftrightarrow \overline{Q})$.

R Never write abusive equivalences !
Each equivalence corresponds to two implications and needs a double reflection !

■ **Example 1.10** For x, y reals : $x = y$ is not equivalent to $x^2 = y^2$. ■

1.2.6 Quantifiers

Definition 1.2.9 — Universal Quantifier. Let $P(x)$ an assertion depending in an element $x \in E$.
We define the assertion

$$\forall x \in E, P(x)$$

as being true when $P(x)$ is true for all x in E. This assertion reads : «For all x in E we have $P(x)$ ».

■ **Example 1.11** $\forall x \in \mathbb{R},\ x^2 \geqslant 0$ is true.
$\forall x \in [-1, 1], x^2 \leqslant 2$ is true.
$\forall x \in \mathbb{R},\ x^2 \leqslant 2$ is false. ■

R Writing $\forall x, P(x)$ is insufficient !
In the Writing « $\forall x \in E, P(x)$ », the letter x has a muted role i.e. which means it can be replaced with any other letter.

Definition 1.2.10 — Existential Quantifier. We define the assertion

$$\exists x \in E, P(x)$$

as true when $P(x)$ is true for at least one x in E. This assertion reads : «There exists an x in E such that $P(x)$ ».

■ **Example 1.12** $\exists x \in \mathbb{R}, x^2 = 1$ is true.
$\exists x \in \mathbb{R}, x^2 < 0$ is false. ■

Definition 1.2.11 We define the assertion $\exists! x \in E, P(x)$ as true when $P(x)$ is true for one, and only one element x in E. This assertion reads : «There exists a unique x in E such that $P(x)$ ».

■ **Example 1.13** $\exists! x \in \mathbb{R}^{+*}, \ln x = 1$ is true (It's Neper's Number).
$\forall x \in \mathbb{R}, \exists! n \in \mathbb{Z}, n \leqslant x < n+1$ is true.
In this last assertion, the integer n appears after x and is therefore likely to depend on x, we often write n_x to emphasize it. This assertion is true and for each x, the integer n introduced is called the integer part of x. ■

R Never switch, without justification the $\forall$ and the $\exists$:
« $\forall x \in \mathbb{R}, \exists n \in \mathbb{Z}, x \leqslant n$ » is true while « $\exists n \in \mathbb{Z}, \forall x \in \mathbb{R}, x \leqslant n$ » is false.
Abusively, we write:

$\forall x \geqslant 0$ instead of $\forall x \in \mathbb{R}^+$,
$\forall 0 \leqslant x \leqslant 1$ instead of $\forall x \in [0,1]$,
$\forall x, y \in E$ instead of $\forall x \in E, \forall y \in E$ or $\forall (x,y) \in E^2$.

Proposition 1.2.7 — Negating Quantifiers.

$$\overline{(\forall x \in E, P(x))} \Leftrightarrow \exists x \in E, \overline{P(x)}$$

$$\overline{(\exists x \in E, P(x))} \Leftrightarrow \forall x \in E, \overline{P(x)}$$

■ **Example 1.14** Let $f : \mathbb{R} \to \mathbb{R}$ a function.
The function f is called the null function : $\forall x \in \mathbb{R}, f(x) = 0$.
f is not the null function : $\exists x \in \mathbb{R}, f(x) \neq 0$.
f cancels out : $\exists x \in \mathbb{R}, f(x) = 0$.
f never cancels out: $\forall x \in \mathbb{R}, f(x) \neq 0$.
f cancels out only once : $\exists! x \in \mathbb{R}, f(x) = 0$.
f takes only positive values : $\forall x \in \mathbb{R}, f(x) \geq 0$.
f takes only positive values in $\mathbb{R}^+$: $\forall x \in \mathbb{R}, f(x) \geq 0 \Rightarrow x \geqslant 0$.
f is constant : $\forall x, y \in \mathbb{R}, f(x) = f(y)$, or even : $\exists C \in \mathbb{R}, \forall x \in \mathbb{R}, f(x) = C$.
f is increasing : $\forall x, y \in \mathbb{R}, x < y \Rightarrow f(x) \leqslant f(y)$.
f takes two-on-two distinct values : $\forall x, y \in \mathbb{R}, x \neq y \Rightarrow f(x) \neq f(y)$, or else
$\forall x, y \in \mathbb{R}, f(x) = f(y) \Rightarrow x = y$. ■

1.3 Reasoning

In this section, we delve into the art of logical thought processes, examining how reasoning forms the bedrock for mathematical analysis and problem-solving. Sharpen your reasoning skills as we unravel the intricacies of deductive and inductive reasoning.

1.3.1 Demonstrate an implication : The direct reasoning and the reasoning by contraposition

Proposition 1.3.1 — Schema. To demonstrate the veracity of an implication $P \Rightarrow R$ there are two methods :

- Direct : We suppose P true and prove that it concludes to the veracity of R.
- By contraposition : we prove that the implication $\overline{R} \Rightarrow \overline{P}$ is true.

■ **Example 1.15** Let us prove that if n is an even integer, then $3n+7$ is an odd integer.
Let n an even integer, then so is $3n$, therefore $3n+7$ is odd. ■

■ **Example 1.16** Let $n \in \mathbb{Z}$, we are looking to prove that, if $5n-7$ is even, then n is odd.
By contraposition, we just have to prove that if n is even, then $5n-7$ is odd.
Indeed, let us suppose that n is even i.e. $\exists k \in \mathbb{Z}, n = 2k$
then $5n-7 = 10k-7 = 2(5k-4)+1$, we deduce that $5n-7$ is odd, hence the result. ■

1.3.2 The deductive reasoning

Proposition 1.3.2 — Schema. When R is a true assertion, and $R \Rightarrow P$ is a true implication, we can affirm that P is true.

R To demonstrate the veracity of an assertion P we can prove that P follows anterior results i.e. to determine a statement R (true) such that $R \Rightarrow P$ is true.
A known true result (like a theorem) can only give another true result. This law is known as « modus ponens ».

■ **Example 1.17** Let us prove that : $\forall x \in \mathbb{R},\ x^2+1>0$.
Let $x \in \mathbb{R}$. We know that $x^2 \geqslant 0$ and $1>0$ and that $a \geqslant 0$ et $b>0 \Rightarrow a+b>0$ so $x^2+1>0$. ■

1.3.3 Reasoning by absurd

Proposition 1.3.3 — Schema. When $\overline{P} \Rightarrow R$ is a true implication, and R is a false assertion, we can affirm that P is true assertion.

R we want to prove that an assertion P is true. We suppose that it's negation $\overline{P}$ is true and we prove that it gives a false assertion (absurd). We conclude that P is true.

■ **Example 1.18** Let us prove that there exists no strict positive real number which is smaller than any other.
Let us suppose by absurd that there exists a real $r>0$ which verifies that property, then $0<r/2<r$.
So $r/2$ is a strict positive real number and is smaller than r, contradiction ! ■

■ **Example 1.19 — Other examples.** The proof of irrationality of $\sqrt{2}$, the infinity of prime numbers, the Impossibility of traveling through time ... ■

1.3.4 Counterexample reasoning

Proposition 1.3.4 — Schema. $\overline{\forall x \in E, P(x)} \ \Leftrightarrow\ \exists x \in E, \overline{P(x)}$.

R This is used to prove that certain affirmations, pretending to a character of generality (that's to say universal propositions) are false.
Having a statement starting by « for all… », to prove that it is false, it is enough to find an element (« there exists… ») that makes the imposed conditions in the hypothsis without verifying the conclusion. It is the data for the counterexample.

■ **Example 1.20** Let us prove that « $\forall x \in \mathbb{R}, \left(x^2-1\right)>0$ » is false.
Indeed, for $x=1$ we have : $\left(x^2-1\right)=(1^2-1)=0$, so 1 is a counterexample. ■

■ **Example 1.21** Let us prove that « for all n natural integer, n^2+n+41 is a prime number » is false.
Indeed : for $n=41 \in \mathbb{N}$, the number n^2+n+41 is not prime as a multiple of 41. ■

1.3.5 Reasoning by case disjunction

Proposition 1.3.5 — Schema. To prove that an assertion P is true, we can determine a statement R such that the implications $R \Rightarrow P$ and $\overline{R} \Rightarrow P$ are true.

R It consists of decomposing the proposition that we are looking to demonstrate into a finite number of cases (sub-propositions) that are verified independently.

■ **Example 1.22** Let us prove that $\forall n \in \mathbb{N}, \dfrac{n(n+1)}{2} \in \mathbb{N}$.
Let $n \in \mathbb{N}$.
If n is even, then we can write $n = 2k$ with $k \in \mathbb{N}$ and then $\dfrac{n(n+1)}{2} = k(2k+1) \in \mathbb{N}$.
If n is odd, then we can write $n = 2k+1$ with $k \in \mathbb{N}$ and then $\dfrac{n(n+1)}{2} = (2k+1)(k+1) \in \mathbb{N}$.
In both cases : $\dfrac{n(n+1)}{2} \in \mathbb{N}$. ■

■ **Example 1.23** The use of the table of truth to demonstrate Morgan's Law. ■

1.3.6 Inductive reasoning

Theorem 1.3.6 Let $n_0 \in \mathbb{N}$ and $P(n)$ an assertion depending on an integer $n \geqslant n_0$.
If
1. $P(n_0)$ is true;
2. $\forall n \geqslant n_0, P(n)$ is true $\Rightarrow P(n+1)$ is true.

then $\forall n \geqslant n_0,\ P(n)$ is true.

■ **Example 1.24** Let us prove that $\forall n \in \mathbb{N}^*, 1+2+3+....+n = \dfrac{n(n+1)}{2}$.
Let's proceed by induction on $n \in \mathbb{N}^*$.
For $n = 1$, we have : $1 = \dfrac{1+1}{2}$.
Let's suppose the statement is true at rank $n \geqslant 1$ and let us prove that it is true at rank $n+1$.

$$1+2+3+....+n+(n+1) = (1+2+3+....+n)+(n+1)$$

By induction hypothesis, we get

$$1+2+3+....+n+(n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}$$

Induction established. ■

R **Classic Errors to avoid commiting :**

- Beleve that the opposite of $x \geqslant 0$ is $x \leqslant 0$. The opposite of $x \geqslant 0$ is $x < 0$.
- To confuse $\Rightarrow$ and $\Leftrightarrow$. An equivalence is incorporated by two implications.
- To forget the use of quantifiers $\forall$ and $\exists$. For example, The sentence $sin(x) \neq x$ does not make sense. Does it mean
  $\forall x \in \mathbb{R},\ sin(x) \neq x$, in which case it is false because $sin(0) = 0$, or does it mean $\exists x \in \mathbb{R}, sin(x) \neq x$ in which case it is true.

In general, all results containing one varaible must be preceded by the appropriate quantifier.

- A sentence of the type « $\forall$ point $M \in$ on the plane, . . . » is wrong, because it mixes two languages. We must either write it « $\forall M \in P$ », or « for all point M on the plane ».
- To think that the sentences « $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}, m > n$ » and « $\exists m \in \mathbb{N}, \forall n \in \mathbb{N}, m > n$» means the same thing, when the first one is true and the second is false ! We can switch quantifiers of the same nature but not of different natures.
- To prove the veracity of an assertion P, never begin by « let us suppose that P is true ...». This is imagination and not logical reasoning !

2. Sets, Relations and Applications

In the realm of mathematics, the study of sets and relations forms a foundational framework with wide-ranging applications. This chapter delves into the fundamental concepts of sets, explores the intricacies of relations, and investigates their diverse applications in various mathematical contexts.

2.1 Introduction to the notion of sets

Sets are a fundamental concept in mathematics, serving as the building blocks of various mathematical structures. This section introduces the basic notions of sets, exploring their definition, representation, and essential properties.

2.1.1 First notions

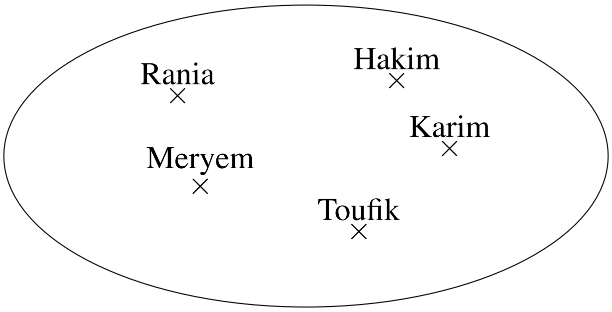
Definition 2.1.1 — Set. A **set** is a collection of distinct objects called **elements**. We can define a set with two methods:

- by **extension**: we list all the elements;
- by **intensive definition or comprehension**: we give a common property that is verified by all the elements of the set.

■ **Example 2.1**

- A set of students:
  - $\{$Hakim;Toufik;Meryem;Rania;Karim$\}$
  - $\{$Class Students with blue eyes$\}$
  - $\{$Teachers of the first year class$\}$
- Classic set of numbers:
  - $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ is the set of natural numbers;
  - $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ is the set of integers;

- $\mathbb{Q} = \{p/q : p \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $q \neq 0\}$ is the set of rational numbers;
- $\mathbb{R}$ the set of real numbers;



■

Without going into the axiomatic, the set notion satisfies certain operating rules:

Proposition 2.1.1 — Main operating rules.
Membership relationship : We write $x \in A$ if the element x is in A.
Distinct objects : We can distinguish two elements and a set cannot contain the same object twice.
The empty set : There exists a set which contains no element, it is the empty set noted $\emptyset$.

2.1.2 Sub Sets

Definition 2.1.2 — Inclusion. The set A is a **subset** of B if all the elements of A are elements of B, in other words

$$x \in A \Longrightarrow x \in B$$

We write $A \subseteq B$ (A **is contained** in B).

■ **Example 2.2**

$$\{0,1,2\} \subseteq \{0,1,2,3\} \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

■

R

$$A = B \text{ if, and only if } A \subseteq B \text{ and } B \subseteq A$$

$$A = B \text{ if, and only if } x \in A \Leftrightarrow x \in B$$

Definition 2.1.3 — Power Set. Let A a set, the **power set of** A written $\mathscr{P}(A)$ is the set of all subsets of A.

R We always have

- $\emptyset \in \mathscr{P}(A)$ because $\emptyset \subseteq A$,
- $A \in \mathscr{P}(A)$ because $A \subseteq A$.

■ **Example 2.3** If $A = \{1,2,3\}$ then

$$\mathscr{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

■

Definition 2.1.4 — Cartesian product of two sets. Let A and B two sets, the **cartesian product of** A **and** B denoted $A \times B$ is the set of couples (a,b) such that $a \in A$ and $b \in B$.

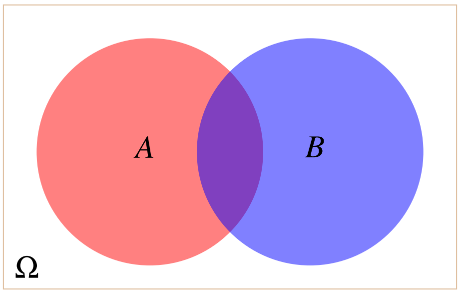
■ **Example 2.4** If $A = \{1,2,3\}$ et $B = \{a,b\}$ then

$$A \times B = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$$

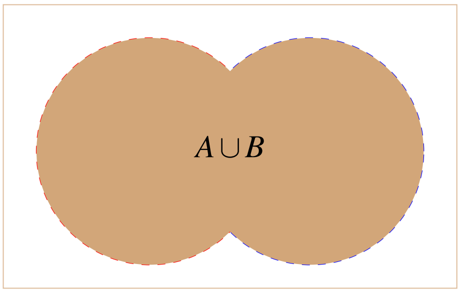
■

2.1.3 Operations on sets

In what follows, we consider :



Definition 2.1.5 — Union. $A \cup B = \{\text{elements of } A \text{ or } B\}$



Proposition 2.1.2 — Properties.

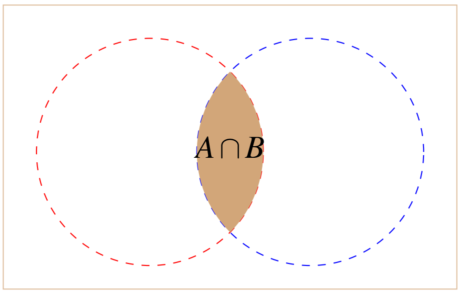
Idempotence: $A \cup A = A$.

Commutativity: $A \cup B = B \cup A$.

Associativity: $A \cup (B \cup C) = (A \cup B) \cup C$.

Neutral element: $A \cup \emptyset = A$.

Definition 2.1.6 — Intersection. $A \cap B = \{\text{elements of } A \text{ and } B\}$

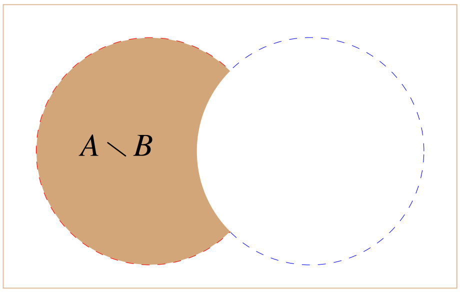


Proposition 2.1.3 — Properties.
Idempotence: $A \cap A = A$.
Commutativity: $A \cap B = B \cap A$.
Associativity: $A \cap (B \cap C) = (A \cap B) \cap C$.
Neutral element: $A \cap \Omega = A$.

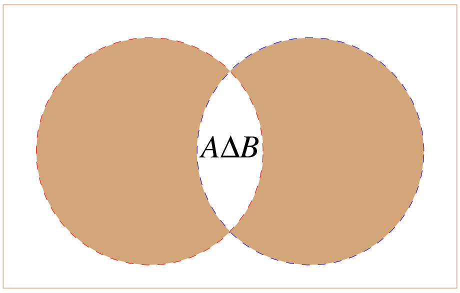
Proposition 2.1.4 — Distributivity.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \text{ and } A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

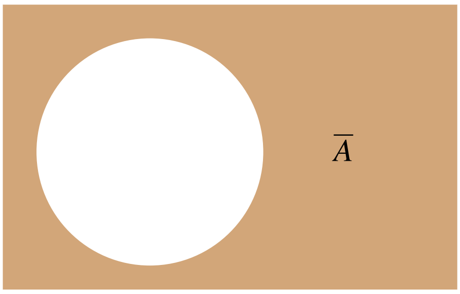
Definition 2.1.7 — Difference Set. $A \setminus B = \{\text{elements of } A \text{ and not of } B\}$



Definition 2.1.8 — Symmetric difference. $A \Delta B = \{\text{elements of } A \cup B \text{ and not of } A \cap B\}$
$= (A \cup B) \setminus (A \cap B)$



Definition 2.1.9 — Complement. $\overline{A} = \Omega \setminus A$



Proposition 2.1.5 — Properties.
Involution: $\overline{\overline{A}} = A$
De Morgan's Law: $\overline{A \cap B} = \overline{A} \cup \overline{B}$ and $\overline{A \cup B} = \overline{A} \cap \overline{B}$

2.2 Relations

This section unravels the intricacies of relations, exploring their definitions, properties, and applications across various mathematical domains.

Definition 2.2.1 — Binary relation. A **Binary relation** $\mathscr{R}$ of a set E to a set F is defined by a set $G_{\mathscr{R}} \subseteq E \times F$.
If $(x,y) \in G_{\mathscr{R}}$, we say that x is **in relation with** y and we write $x\mathscr{R}y$.
If $E = F$ we say that $\mathscr{R}$ is an **internal relation** in E.

■ **Example 2.5** Let $A = \{a,b,c,d,e\}$ the set of of students and $B = \{Math, Info, Ang, Phys\}$ the set of lessons. We can define the following relations:

- $\mathscr{R}$ describes if a student follows a lesson regularly:

$$G_{\mathscr{R}} = \{(a,Math),(a,Phys),(b,Info),(c,Ang),(d,Ang),(e,Math),(e,Ang)\}$$

- The relation $\mathscr{S}$ that describes if a student has lent an object to another student is defined by

$$G_{\mathscr{S}} = \{(b;a);(a;a);(c;a);(a;d);(d;c)\}$$

■

Definition 2.2.2 — Functional Relation. A function $f : E \to F$ associates to each element of E at most one element of F. We can define the relation $\mathscr{R}_f$ defined by the graph

$$G_{\mathscr{R}_f} = \{(x, f(x)) : x \in E\} \subseteq E \times F.$$

Reciprocally, for a relation $\mathscr{R}$ such that for all $x \in E$ there is at most one $y \in F$ verifying $x\mathscr{R}y$ then we can associate a function f to it, such that $f(x) = y$ if and only if $x\mathscr{R}y$. We say that $\mathscr{R}$ is a **functional relation**.

Definition 2.2.3 — Reflexivity. A relation $\mathscr{R}$ is **reflexive** if for all $x \in E$ we have $x\mathscr{R}x$.

■ **Example 2.6** Regardless of the set, the equality relation $=$ is reflexive. In $\mathbb{N}$, the $\leq$ relation is reflexive, but $<$ is not. ■

Definition 2.2.4 — Symmetry. A relation $\mathscr{R}$ is **symmetric** if for all $x, y \in E$ we have $x\mathscr{R}y$ if and only if $y\mathscr{R}x$.

■ **Example 2.7** Regardless of the set, the equality relation $=$ is symmetric.
In $\mathbb{N}$, the relation $\leq$ is not symmetric. ■

Definition 2.2.5 — Transitivity. A relation $\mathscr{R}$ is **transitive** if for all $x, y, z \in E$ such that $x\mathscr{R}y$ and $y\mathscr{R}z$ then necessarily we have $x\mathscr{R}z$.

■ **Example 2.8** Regardless of the set, the equality relation $=$ is transitive.
In $\mathbb{N}$, the relation $\leq$ is transitive.
The relation "is the father of" is not transitive. ■

Definition 2.2.6 — Anti-symmetry. A relation $\mathscr{R}$ is **anti-symmetric** if for all $x, y \in E$ verifying $x\mathscr{R}y$ and $y\mathscr{R}x$ then we have $x = y$.

■ **Example 2.9** In $\mathbb{N}$, the relation $\leq$ is anti-symmetric. ■

2.2.1 Equivalence Relations

Definition 2.2.7 — Equivalence Relation. A binary relation defined by a unique set E is an **equivalence relation** if it is reflexive, symmetric and transitive.

■ **Example 2.10**

- In any set, the relation $=$ is an equivalence relation.
- In the set of persons, the relation "has the same age as" is an equivalence relation. Persons related belong to the same age range.
- In the set of triangles, the relation "has the same angles as" is an equivalence relation. Triangles linked by the relation are said to be similar.
- The relation $\mathscr{R}$ defined in $\mathbb{R} \setminus \{0\}$ by $x\mathscr{R}y$ if and only if $xy > 0$ is an equivalence relation.

■

Definition 2.2.8 — Equivalence class. Let $\mathscr{R}$ an equivalence relation in a set E. The **equivalence class** of an element x, written $\mathbf{Cl}(x)$, is the set of elements of E which are related to x. In other words

$$\mathbf{Cl}(x) = \{y \in E : x\mathscr{R}y\}.$$

Proposition 2.2.1 An equivalence class is never empty.
The intersection of two distinct equivalence classes is empty.

■ **Example 2.11** For the relation $\mathscr{R}$ defined in $\mathbb{R} \setminus \{0\}$ by $x\mathscr{R}y \Leftrightarrow xy > 0$,

$$\mathbf{Cl}(3) = \{y \in \mathbb{R}^* : 3\mathscr{R}y\} = \{y \in \mathbb{R}^* : 3y > 0\} =]0, +\infty[$$

$$\mathbf{Cl}(-2.3) =]-\infty, 0[$$

■

■ **Example 2.12** Quotient set Let E a set endowed with an equivalence relation $\mathscr{R}$. The **Quotient Set** is the set of equivalence classes of all the elements of E. We write it $E/\mathscr{R}$. ■

Theorem 2.2.2 Given an equivalence relation $\mathscr{R}$ in E, the following function is a surjective function:

$$\begin{array}{rccc} f: & E & \longrightarrow & E/\mathscr{R} \\ & x & \longmapsto & \mathbf{Cl}(x) \end{array}$$

■ **Example 2.13** For the relation $\mathscr{R}$ defined in $\mathbb{R} \setminus \{0\}$ by $x\mathscr{R}y \Leftrightarrow xy > 0$,

$$E/\mathscr{R} = \{]-\infty, 0[,]0, +\infty[\}.$$

■

2.2.2 Order Relations

Definition 2.2.9 — Order Relation. A binary relation $\preceq$ in a set E is an **order relation** if it is reflexive, transitive and anti-symmetric. In other words:
$\preceq$ **reflexive:** we have $x \preceq x$ for all $x \in E$.
$\preceq$ **transitive:** if $x \preceq y$ and $y \preceq z$ then $x \preceq z$.
$\preceq$ **anti-symmetric:** if $x \preceq y$ and $y \preceq x$ then $x = y$.
An order is **total** if for all elements $x, y \in E$ we have $x \preceq y$ or $y \preceq x$. An order is said to be **partial** to highlight that we do not necessarily have that property.

■ **Example 2.14 — Examples of orders in numbers.**

- $\leq$ and $\geq$ are total order relations in $\mathbb{N}$ which extends to $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$.
- $<$ and $>$ are not order relations in $\mathbb{N}$.
- In $\mathbb{N}^*$ the relation a divides b, written $a|b$, is an order relation but is not total. We recall that a divides b if there exists $k \in \mathbb{N}^*$ such that $b = ak$.
  Reflexivity : we have $x|x$ for all $x \in \mathbb{N}^*$.
  Transitivity : if x divides y (which means if there exists k such that $y = kx$) and y divides z (which means if there exists k' such that $z = k'y$) then $z = kk'x$ so x divides z.
  Anti-symmetry : if x divides y and y divides x then $x = y$ (because, in this case $x \leq y$ and $y \leq x$).

  The relation divide is a partial order relation in $\mathbb{N}^*$, since there are elements (2 and 5 for example) which are not comparable.

■

■ **Example 2.15 — Example of an order relation in a set of a set.** Let E a set. The inclusion, written $\subseteq$, is an order relation in the set $\mathscr{P}(E)$ which is not total.
$\subseteq$ **reflexive:** we have $A \subseteq A$ for all $A \in \mathscr{P}(E)$.
$\subseteq$ **transitive:** if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.
$\subseteq$ **anti-symmetric:** if $A \subseteq B$ and $B \subseteq A$ then $A = B$.
In case where E contains at least two elements i.e.

$$E = \{a, b, ...\}$$

we will have

$$\mathscr{P}(E) = \{\emptyset, \{a\}, \{b\}, \ldots.., E\}$$

Since there are sets that are not comparable (none is included in the other) such as $\{a\}$ and $\{b\}$, the order is partial. ■

2.3 Functions and Mappings

This section delves into the definitions, properties, and applications of functions, providing a comprehensive understanding of how mappings capture essential relationships between mathematical objects.

Definition 2.3.1 — Function. A **function** $f : E \longrightarrow F$ (from E to F) is defined by a subset of $G_f \subseteq E \times F$ such that for all $x \in E$, there exists at most one $y \in F$ such that $(x, y) \in G_f$, we write $y = f(x)$.

■ **Example 2.16** Let $E = \{1,2,3,4\}$ and $F = \{a,b,c\}$. We define the function f by the graph:

$$G_f = \{(1,a),(2,c),(4,a)\} \subset E \times F$$

In other words,

$$\begin{array}{rcl} f \quad : E & \longrightarrow & F \\ 1 & \longmapsto & a \\ 2 & \longmapsto & c \\ 4 & \longmapsto & a \end{array}$$

■

■ **Example 2.17** $H = \{(1,a),(2,c),(4,a),(1,b)\} \subset E \times F$ is not the graph of a function. ■

Definition 2.3.2 — Image set. Let $f : E \longrightarrow F$ a function from E to F.
Image: $f(x)$ is the **image** of x
Image set of $A \subset E$:

$$\begin{aligned} f(A) &= \{y \in F \text{ such that } \exists x \in A \text{ which verifies } f(x) = y\} \\ &= \{y \in F \text{ such that } \exists x \in A \text{ which verifies } (x,y) \in G_f\} \end{aligned}$$

Image set of f:

$$\mathrm{Im}(f) = f(E) = \{y \in F :\ \exists x \in E \text{ such that } f(x) = y\}$$

■ **Example 2.18** Let $E = \{1,2,3,4\}$ and $F = \{a,b,c\}$ and $f : E \to F$ defined by $G_f = \{(1,a),(2,c),(4,a)\} \subset E \times F$. We have:

$$f(\{1\}) = \{a\} \qquad f(\{1,4\}) = \{a\} \qquad f(\{3\}) = \emptyset \qquad f(\{1,2,3\}) = \{a,c\}$$

$$\mathrm{Im}(f) = \{a,c\}$$

■

Definition 2.3.3 — Preimage. Let $f : E \longrightarrow F$ a function from E to F.
Antecedent: x is the **antecedent** of y if $y = f(x)$
Preimage of $B \subset F$:

$$\begin{aligned} f^{-1}(B) &= \{x \in E \text{ such that } \exists y \in B \text{ verifying } f(x) = y\} \\ &= \{x \in E \text{ such that } \exists y \in B \text{ verifying } (x,y) \in G_f\} \end{aligned}$$

Domain of definition of f:

$$\mathrm{Dom}(f) = f^{-1}(F) = \{x \in E :\ \exists y \in F \text{ such that } f(x) = y\}$$

■ **Example 2.19** Let $E = \{1,2,3,4\}$ and $F = \{a,b,c\}$ and $f : E \to F$ defined by $G_f = \{(1,a),(2,c),(4,a)\} \subset E \times F$. We have:

$$f^{-1}(\{a\}) = \{1,4\} \qquad f^{-1}(\{a,c\}) = \{1,2,4\} \qquad f^{-1}(\emptyset) = \emptyset \qquad f^{-1}(\{b\}) = \emptyset$$

$$\mathrm{Dom}(f) = \{1,2,4\}$$

■

Definition 2.3.4 — Mapping. A function $f : E \to F$ is a mapping if $\mathrm{Dom}(f) = E$.

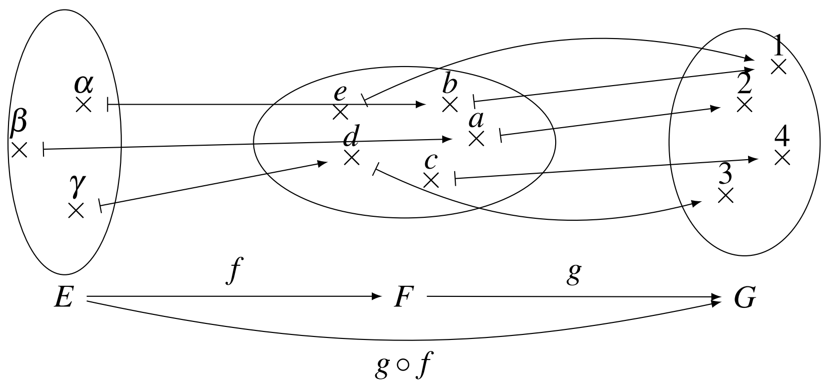
■ **Example 2.20**

- Let $E = \{1,2,3,4\}$ and $F = \{a,b,c\}$.
  The graph $G = \{(1,a),(2,c),(4,a)\} \subset E \times F$ defines a function from E to F but not a mapping.
- Let $E' = \{1,2,4\}$ and $F = \{a,b,c\}$.
  The graph $G = \{(1,a),(2,c),(4,a)\} \subset E' \times F$ defines a function from E' to F which is a mapping from E' to F.

■

Definition 2.3.5 — Composition. The **function composition** from $f : E \to F$ by $g : F \to G$ is defined by

$$g \circ f(x) = g(f(x))$$
$$\mathrm{Dom}(g \circ f) = \{x \in \mathrm{Dom}(f) : f(x) \in \mathrm{Dom}(g)\}$$



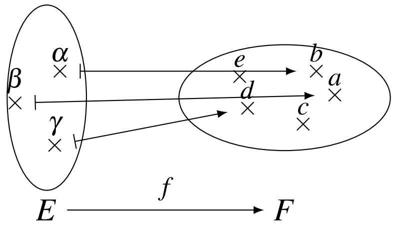
Proposition 2.3.1 — Properties.

- In general $f \circ g \neq g \circ f$.
- Associativity: $(f \circ g) \circ h = f \circ (g \circ h)$.

Definition 2.3.6 — Injective function. $f : E \to F$ is **injective** (one to one) if all $y \in F$ admits at most one antecedent, that's to say : it never takes two same values.

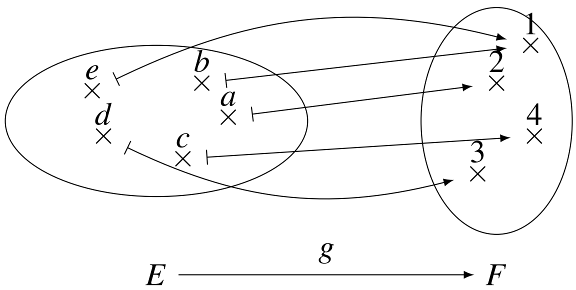
In other words: $\forall x_1, x_2 \in E$ we have $f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$
or : $\forall x_1, x_2 \in E$ we have $x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$

■ **Example 2.21** In case where a group of tourists must be housed in a hostel. The re-partition of the tourists in the chambers of the hostel can be represented by a mapping of the set X of the tourists to the set Y of the chambers.
The tourists wishes that the mapping be **injective**, that's to say that each one of them gets an individual chamber. ■



Definition 2.3.7 — Surjective Function. $f : E \to F$ is **surjective** (Onto) if all $y \in F$ admits at least one antecedent, that's to say that f takes all the images on F.

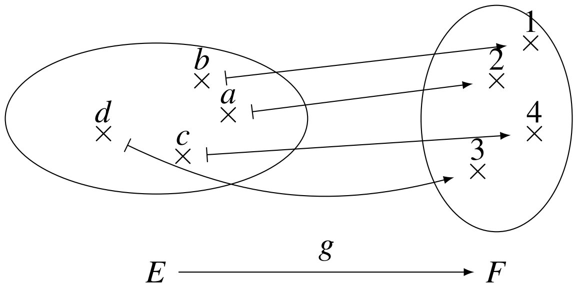
In other words: $\forall y \in F, \exists x \in E, y = f(x)$
Or : $\mathrm{Im}(f) = f(E) = F$.



■ **Example 2.22** Next to the previous example, the hotelier wishes that the mapping be **surjective**, i.e. so that each chamber be occupied. ■

Definition 2.3.8 — Bijective mappings. $f : E \to F$ is a **bijective** mapping if all $y \in F$ admit exactly one antecedent.

In other words, f is an injective and a surjective mapping.



■ **Example 2.23** Note that in our example, it will be possible to share out the tourists in a manner that makes only one tourist occupy a chamber, and that all the chamber be occupied : we then say that the mapping is injective and surjective at the same time, so it is **bijective**. ■

Proposition 2.3.2 — Reciprocal mapping. The mapping $f : E \to F$ is bijective if and only if there exists a mapping $g : F \to E$ such that $f \circ g = \mathrm{Id}_F$ and $g \circ f = \mathrm{Id}_E$.
If f is bijective, the mapping g is unique, it is the **Reciprocal mapping** of the mapping f, noted f^{-1}.

Proposition 2.3.3 — The composite of two bijections. Let $f : E \to F$ and $g : F \to G$ two bijective mappings. The composite $g \circ f$ is bijective and

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

■ **Example 2.24**

- Let ut consider the mappings $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = 3x - 1$. This mapping is injective, since for all $x_1, x_2 \in \mathbb{R}$, $f(x_1) = f(x_2) \implies 3x_1 - 1 = 3x_2 - 1 \implies x_1 = x_2$.
  f is also surjective, indeed : for $y \in \mathbb{R}$, $y = f(x) \implies y = 3x - 1 \implies x = \dfrac{y+1}{3} \in \mathbb{R}$ and so, for all $y \in \mathbb{R}$, $\exists x = \dfrac{y+1}{3} \in \mathbb{R}$ such that $y = f(x)$.
  We conclude that f is bijective.
- The mapping $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $g(x) = x^2$ is not injective, because (for example) $g(1) = g(-1) = 1$.
  g is not surjective either, because -1 has no antecedent by g or the equation $g(x) = -1$ has no solutions in $\mathbb{R}$.

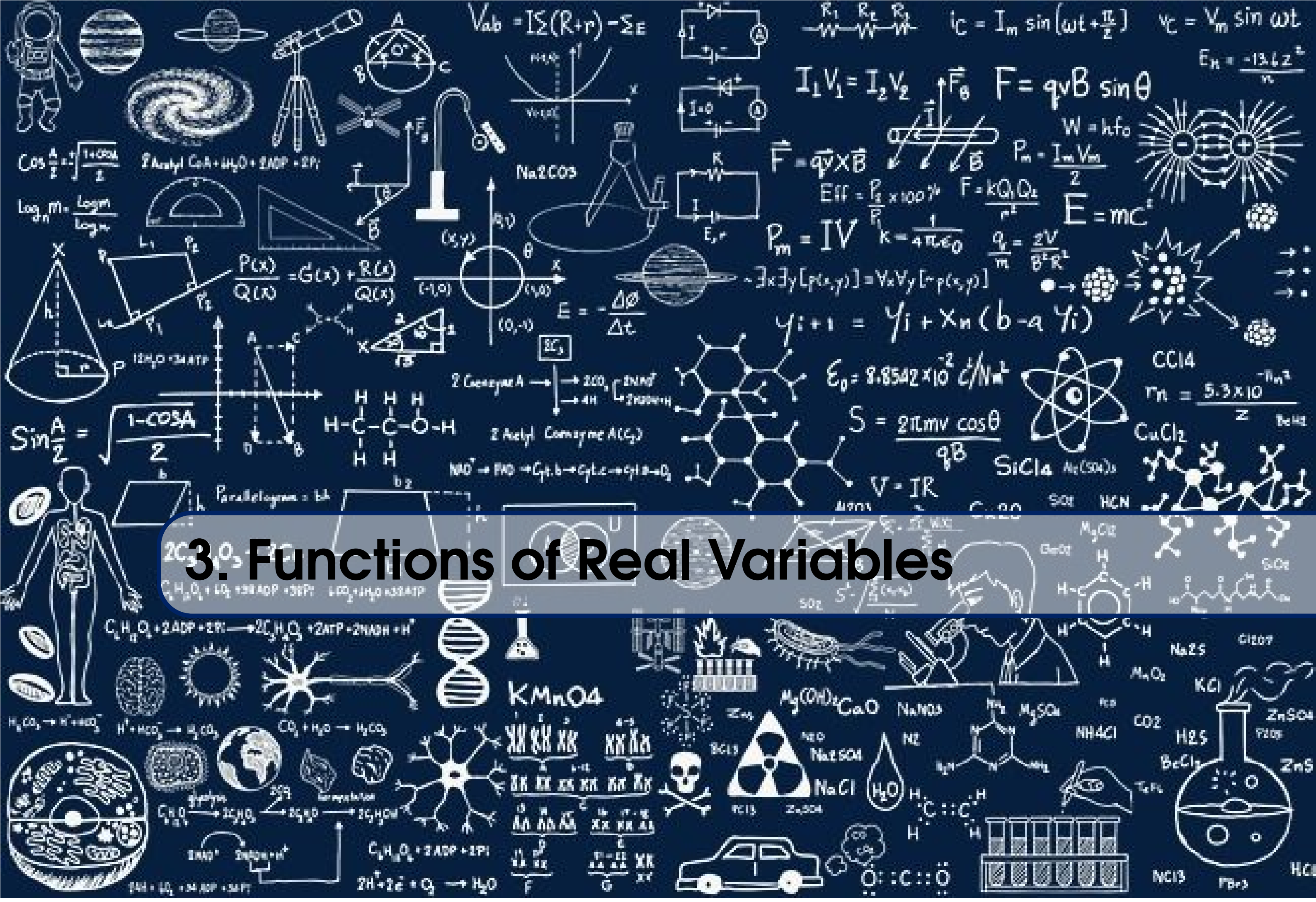
■

Definition 2.3.9 — Increasing and decreasing functions. Let A and B two set endowed respectively with the order relations $\preceq_A$ and $\preceq_B$ and $f : A \longrightarrow B$ a mapping. We say that

- f is **increasing** if $x \preceq_A y$ then $f(x) \preceq_B f(y)$.
- f is **decreasing** if $x \preceq_A y$ then $f(y) \preceq_B f(x)$.
- f is **strictly increasing** if $x \preceq_A y$ and $x \neq y$ then $f(x) \preceq_B f(y)$ and $f(x) \neq f(y)$.
- f is **strictly decreasing** if $x \preceq_A y$ and $x \neq y$ then $f(y) \preceq_B f(x)$ and $f(x) \neq f(y)$.

Proposition 2.3.4 A strictly increasing or decreasing mapping with an endowed total order starting space is injective.

■ **Example 2.25** A mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$ strictly monotone is injective. ■



3. Functions of Real Variables

Immerse yourself in the study of *Functions of Real Variables*, a foundational topic in mathematical analysis. This chapter explores the properties, behaviors, and applications of functions that depend on real variables. Delve into the intricacies of real-valued functions and their significance in understanding the dynamics of mathematical relationships.

3.1 Functions

This section delves into the essential concepts of functions, examining their definitions, properties, and varied applications.

3.1.1 Definitions

Definition 3.1.1 — Definition of a real function. A function of a real variable with multiple real values is a mapping $f : \mathscr{D} \longrightarrow \mathbb{R}$, where $\mathscr{D}$ is a set of $\mathbb{R}$. In general, $\mathscr{D}$ is an interval or a reunion of intervals. We call $\mathscr{D}$ the domain of definition of the function f.

■ **Example 3.1** The inverse function :

$$\begin{array}{rcl} g :]-\infty,0[\cup]0,+\infty[& \longrightarrow & \mathbb{R} \\ x & \longrightarrow & \dfrac{1}{x}. \end{array}$$

■

Definition 3.1.2 We call graph (or representative curve) of a function $f : \mathscr{D} \longrightarrow \mathbb{R}$, the set

$$\Gamma_f = \{M(x,y) \; / \; x \in \mathscr{D}, \; y = f(x)\}$$

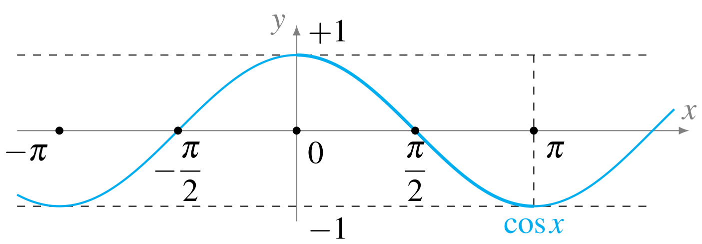
Definition 3.1.3 — Parity of a function. A function $f : \mathscr{D} \longrightarrow \mathbb{R}$ is said **even** (resp. **odd**) if :

1. The domain $\mathscr{D}$ is symmetric compared to 0 i.e. : $\forall x \in \mathscr{D}, \; -x \in \mathscr{D}$.
2. $\forall x \in \mathscr{D}, f(-x) = f(x)$. (resp. $f(-x) = -f(x)$).

Definition 3.1.4 — Periodicity. A function $f : \mathscr{D} \longrightarrow \mathbb{R}$ is said T periodic if :

1. The domain $\mathscr{D}$ is T periodic i.e. : $\forall x \in \mathscr{D}, \; x + T \in \mathscr{D}$
2. $\forall x \in \mathscr{D}, f(x + \mathrm{T}) = f(x)$.

■ **Example 3.2** the function $\cos : \mathbb{R} \longrightarrow [-1, 1]$ is a real function, even and 2π periodic.



■

Definition 3.1.5 — Operations on functions. Let $f : \mathscr{D} \to \mathbb{R}$ and $g : \mathscr{D} \to \mathbb{R}$ two functions defined by the same set $\mathscr{D}$ of $\mathbb{R}$. We can then define the following functions :

- The sum of f and g is the function $f + g : \mathscr{D} \to \mathbb{R}$ defined by $(f+g)(x) = f(x) + g(x)$ for all $x \in \mathscr{D}$.
- The product of f and g is the function $f \times g : \mathscr{D} \to \mathbb{R}$ defined by $(f \times g)(x) = f(x) \times g(x)$ for all $x \in \mathscr{D}$.
- The multiplication by a scalar $\lambda \in \mathbb{R}$ of f is the function $\lambda \cdot f : \mathscr{D} \to \mathbb{R}$ defined by $(\lambda \cdot f)(x) = \lambda \cdot f(x)$ for all $x \in \mathscr{D}$.

■ **Example 3.3** the function $x \to x \ln x - 3x^2 + 2 \sin x$ is well defined in $]0, +\infty[$. ■

Definition 3.1.6 — Upper bounded, lower bounded, Bounded, Constant functions.... Let $f : \mathscr{D} \to \mathbb{R}$ and $g : \mathscr{D} \to \mathbb{R}$ two functions. so :

- f is said **upper bounded** in $\mathscr{D}$ if : $\exists M \in \mathbb{R}, \;\; \forall x \in \mathscr{D}, \; f(x) \leqslant M$.
- f is said **lower bounded** in $\mathscr{D}$ if : $\exists m \in \mathbb{R}, \;\; \forall x \in \mathscr{D}, \; f(x) \geqslant m$.
- f is **bounded** in $\mathscr{D}$ if f is lower and upper bounded at the same time in $\mathscr{D}$, that's to say if : $\exists M \in \mathbb{R}, \forall x \in \mathscr{D}, \;\; |f(x)| \leqslant M$.
- $f \geqslant g$ if $\forall x \in \mathscr{D} \;\; f(x) \geqslant g(x)$.
- $f \geqslant 0$ if $\forall x \in \mathscr{D} \;\; f(x) \geqslant 0$.
- $f > 0$ if $\forall x \in \mathscr{D} \;\; f(x) > 0$.
- f is said **constant** in $\mathscr{D}$ if : $\exists a \in \mathbb{R}, \;\; \forall x \in \mathscr{D}, \; f(x) = a$.
- f is said **null** in $\mathscr{D}$ if : $\forall x \in \mathscr{D}, \;\; f(x) = 0$.

■ **Example 3.4** The functions cos and sin are bounded, the function $x \to e^x$ is lower bounded, the function $x \to \ln x$ is neither lower nor upper bounded. ■

Definition 3.1.7 — Variations of a function. Let $f : \mathscr{D} \to \mathbb{R}$ a function. We say that :

- f is **increasing** in $\mathscr{D}$ if : $\forall x, y \in \mathscr{D}, \quad x < y \Rightarrow f(x) \leqslant f(y)$.
- f is **strictly increasing** in $\mathscr{D}$ if : $\forall x, y \in \mathscr{D}, \quad x < y \Rightarrow f(x) < f(y)$.
- f is **decreasing** in $\mathscr{D}$ if : $\forall x, y \in \mathscr{D}, \quad x < y \Rightarrow f(x) \geqslant f(y)$.
- f is **strictly decreasing** in $\mathscr{D}$ if : $\forall x, y \in \mathscr{D}, \quad x < y \Rightarrow f(x) > f(y)$.
- f is **monotone** (resp. **strictly monotone**) in $\mathscr{D}$ if f is increasing or decreasing (resp. strictly increasing or strictly decreasing) in $\mathscr{D}$.

3.2 Limits

In this section, we explore the foundational principles behind limits, understanding how they define the behavior of functions as they approach specific points.

Definition 3.2.1 — Finite Limit on one point. Let $f : I \to \mathbb{R}$ a function defined in an interval I of $\mathbb{R}$, $x_0 \in \mathbb{R}$ a point of I or an extremity of I and $\ell \in \mathbb{R}$.
We say that f has for limit ℓ in x_0 if

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in I, \quad |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \varepsilon.$$

We also say that $f(x)$ tends to ℓ when x tends to x_0.
We then write $\lim_{x \to x_0} f(x) = \ell$ or $\lim_{x_0} f = \ell$.
We say that f converges on x_0 if there exists $\ell \in \mathbb{R}$ (a finite real) such that $\lim_{x \to x_0} f(x) = \ell$.
Otherwise, we say that f diverges on x_0.

R The inequality $|x - x_0| < \delta$ is equivalent to $x \in]x_0 - \delta, x_0 + \delta[$.
The inequality $|f(x) - \ell| < \varepsilon$ is equivalent to $f(x) \in]\ell - \varepsilon, \ell + \varepsilon[$.
This definition means that $f(x)$ gets closer as much as we want to ℓ when x is getting closer sufficiently to x_0.

■ **Example 3.5** Using the definition of limit, we show that $\lim_{x \to 1}(2x - 3) = -1$
i.e. let us prove that

$$\forall \varepsilon > 0, \quad \exists \delta > 0, \quad \forall x \in \mathbb{R} : \quad |x - 1| < \delta \Rightarrow |(2x - 3) - (-1)| < \varepsilon.$$

Let $\varepsilon > 0$. We have

$$|(2x - 3) - (-1)| < \varepsilon \Leftrightarrow |2x - 2| < \varepsilon \Leftrightarrow 2|x - 1| < \varepsilon$$

It is sufficient to take $\delta = \varepsilon/2$ (Or whatever δ smaller than $\varepsilon/2$).
So,

$$\forall \varepsilon > 0, \quad \exists \delta = \varepsilon/2, \quad \forall x \in \mathbb{R} : \quad |x - 1| < \delta \Rightarrow |(2x - 3) - (-1)| < \varepsilon.$$

■

Definition 3.2.2 — Infinite limit on one point. Let f a function defined in a set of the form $I =]a, x_0[\cup]x_0, b[$.
We say that f has for limit $+\infty$ on x_0 if

$$\forall A > 0, \quad \exists \delta > 0, \quad \forall x \in I \ : \quad |x - x_0| < \delta \Rightarrow f(x) > A$$

We then write $\lim_{x \to x_0} f(x) = +\infty$.

We say that f has for limit $-\infty$ in x_0 if

$$\forall A > 0, \quad \exists \delta > 0, \quad \forall x \in I \ : \quad |x - x_0| < \delta \Rightarrow f(x) < -A$$

We then write $\lim_{x \to x_0} f(x) = -\infty$.

R We can easily deduce the definition of other cases such as :

$$\lim_{x \to +\infty} f(x) = \ell \Leftrightarrow \forall \varepsilon > 0, \quad \exists A > 0, \quad \forall x \in I \ : \quad x > A \Rightarrow |f(x) - \ell| < \varepsilon$$

$$\lim_{x \to -\infty} f(x) = +\infty \Leftrightarrow \forall A > 0, \quad \exists B > 0, \quad \forall x \in I \ : \quad x < -B \Rightarrow f(x) > A.$$

Definition 3.2.3 — Left and right limit. Let f a function defined on a set of the form $]a, x_0[\cup]x_0, b[$.
Saying that f admits a right limit $\ell \in \mathbb{R}$ in x_0 means :

$$\forall \varepsilon > 0, \quad \exists \delta > 0 : \quad x_0 < x < x_0 + \delta \Rightarrow |f(x) - \ell| < \varepsilon.$$

Saying that f admits a left limit $\ell \in \mathbb{R}$ in x_0 means :

$$\forall \varepsilon > 0, \quad \exists \delta > 0 : \quad x_0 - \delta < x < x_0 \Rightarrow |f(x) - \ell| < \varepsilon.$$

We write $\lim_{x \to x_0^+} f(x)$ for the right limit and $\lim_{x \to x_0^-} f(x)$ for the left limit.
If the function f has a limit in x_0, then its left and right limits in x_0 coincide and their values are $\lim_{x_0} f$.
Reciprocally, if f has a left and right limit in x_0 and if its limits values are $f(x_0)$ (if f is well defined in x_0) then f admits a limit in x_0.

■ **Example 3.6** Let E the floor set function, since $\lim_{x \to 2^+} E(x) = 2$ et $\lim_{x \to 2^-} E(x) = 1$, this function diverges in 2. However, it converges in 2.4. ■

R We put $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, \ -\infty\}$.
We write $f \underset{x_0}{\longrightarrow} \ell^+$ (resp. $f \underset{x_0}{\longrightarrow} \ell^-$) to mean that $f \underset{x_0}{\longrightarrow} \ell$ and $f(x) > \ell$ (resp. $f(x) < \ell$) when x is sufficiently close to x_0.

Theorem 3.2.1 — Operations on limits.

- If $f \underset{a}{\to} \ell \in \overline{\mathbb{R}}$ and $g \underset{a}{\to} \ell' \in \overline{\mathbb{R}}$ and $\ell + \ell'$ is defined in $\overline{\mathbb{R}}$ then $f+g \underset{a}{\to} \ell + \ell'$.
- If $f \underset{a}{\to} \ell \in \overline{\mathbb{R}}$ and $g \underset{a}{\to} \ell' \in \overline{\mathbb{R}}$ and $\ell\ell'$ is defined in $\overline{\mathbb{R}}$ then $fg \underset{a}{\to} \ell\ell'$.
- If $f \underset{a}{\to} \ell \in \mathbb{R}^*$ then $1/f \underset{a}{\to} 1/\ell$.
- If $f \underset{a}{\to} 0^+$ (resp. $f \underset{a}{\to} 0^-$) then $1/f \underset{a}{\to} +\infty$ (resp. $1/f \underset{a}{\to} -\infty$).
- If $f \underset{a}{\to} +\infty$ (resp. $f \underset{a}{\to} -\infty$) then $1/f \underset{a}{\to} 0^+$ (resp. $1/f \underset{a}{\to} 0^-$).
- If $Im(f) \subset D_g$, $f \underset{a}{\to} b \in \overline{\mathbb{R}}$ and $g \underset{b}{\to} \ell \in \overline{\mathbb{R}}$ then $g \circ f \underset{a}{\to} \ell$.

Reminder : Let us recall that some forms are undetermined which are the following $+\infty - \infty$ and $0 \times \infty$.

Theorem 3.2.2 — Squeeze theorem. We suppose that at the neighborhood of a (when x is close to a)

$$g(x) \leqslant f(x) \leqslant h(x)$$

If $g \underset{a}{\to} \ell \in \mathbb{R}$ and $h \underset{a}{\to} \ell$ then $f \underset{a}{\to} \ell$

Example 3.7 Let us study $\lim\limits_{x \to +\infty} \dfrac{E(x)}{x}$

We have $x - 1 \leqslant E(x) \leqslant x$, so $\quad 1 - \dfrac{1}{x} \leqslant \dfrac{E(x)}{x} \leqslant 1$.

Then $\lim\limits_{x \to +\infty} \left(1 - \dfrac{1}{x}\right) = 1$ so by framing $\lim\limits_{x \to +\infty} \dfrac{E(x)}{x} = 1$. ■

Proposition 3.2.3 If f is bounded in the vicinity of a and if $g \underset{a}{\to} 0$ then $fg \underset{a}{\to} 0$.

Example 3.8 $\lim\limits_{x \to 0} x \cos(1/x) = 0$. ■

Theorem 3.2.4 — Obtention of infinite limits by comparison. Let us suppose that $f(x) \leqslant g(x)$ at the neighborhood of a.
If $f \underset{a}{\to} +\infty$ then $g \underset{a}{\to} +\infty$.
If $g \underset{a}{\to} -\infty$ then $f \underset{a}{\to} -\infty$.

Example 3.9 When $x \to +\infty$ we have

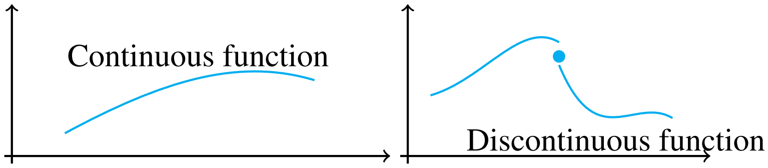
$$x^2 + x \sin x \geq x^2 - x$$

So $x^2 - x \xrightarrow[x \to +\infty]{} +\infty$ and by comparison $x^2 + x \sin x \xrightarrow[x \to +\infty]{} +\infty$. ■

3.3 Continuity

Definition 3.3.1 — Continuity in one point. Let a a real and f a real function defined in $\mathscr{D}$ a set of $\mathbb{R}$ containing a.
f is continuous in a if, and only if $f(x) \xrightarrow[x \to a]{} f(a)$.
In the opposite case, we say that f is discontinuous in a.

Definition 3.3.2 — Continuity in a set. We say that f is continuous if f is continuous on each $a \in \mathscr{D}$. We write $\mathscr{C}(\mathscr{D}, \mathbb{R})$ the set of real functions defined and continuous in $\mathscr{D}$.



■ **Example 3.10** Constant functions are continuous. The functions $x \to x$ and $x \to |x|$ are continuous in $\mathbb{R}$. Generally speaking, The usual functions : sin, cos, tan, exp, ln, $\sqrt{x}$ .... are continuous in their domain of definition. ■

Theorem 3.3.1 — Operations on continuous functions. Let $f, g : \mathscr{D} \to \mathbb{R}$ and $\lambda \in \mathbb{R}$. If f and g are continuous then so are λf, $f + g$ and fg.
In addition, if g never cancels then $\dfrac{f}{g}$ is continuous.

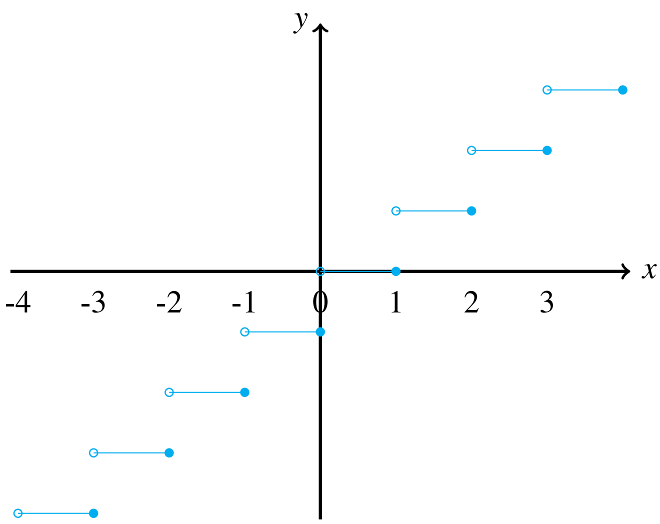
■ **Example 3.11** By the sum and product of continuous functions, the polynomial functions are continuous in $\mathbb{R}$. Compared with the continuous functions, the rational functions are continuous where they are defined. ■

Theorem 3.3.2 Let $f : \mathscr{D} \to \mathbb{R}$ and $g : \mathscr{D}' \to \mathbb{R}$ such that $f(\mathscr{D}) \subset \mathscr{D}'$.
If f and g are continuous then so is $g \circ f$.

■ **Example 3.12** The function $x \to \dfrac{\sqrt{x^2+1}}{\sin^2(3x)+2}$ is continuous in $\mathbb{R}$. ■

Definition 3.3.3 — Left and Right Continuity. Let $f : \mathscr{D} \to \mathbb{R}$ and $a \in \mathscr{D}$.
If f is defined right to a, we say that f is right continuous in a if $f \xrightarrow[x \to a^+]{} f(a)$.
If f is defined left to a, we say that f is left continuous in a if $f \xrightarrow[x \to a^-]{} f(a)$.

■ **Example 3.13** The floor function $x \to E(x)$ is right continuous in all $a \in \mathbb{R}$.



■

Proposition 3.3.3 Let $f : \mathscr{D} \to \mathbb{R}$ and a a point of $\mathscr{D}$ such that f is defined left and right to a. We have an equivalence between:

- f is continuous in a.
- f is left and right continuous in a.

R This is useful when trying to obtain the continuity of a function defined by an alternative.

■ **Example 3.14** The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x^2, & \text{otherwise} \end{cases}$$

It is evident that f is continuous in $]0,+\infty[$ and $]-\infty,0[$, what is left is the study at $a = 0$.
When $x \to 0^+$, $f(x) = x^2 \to 0 = f(0)$ and when $x \to 0^-$, $f(x) = -x^2 \to 0 = f(0)$.
By left and right continuity on 0, f is continuous in 0 and subsequently on $\mathbb{R}$. ■

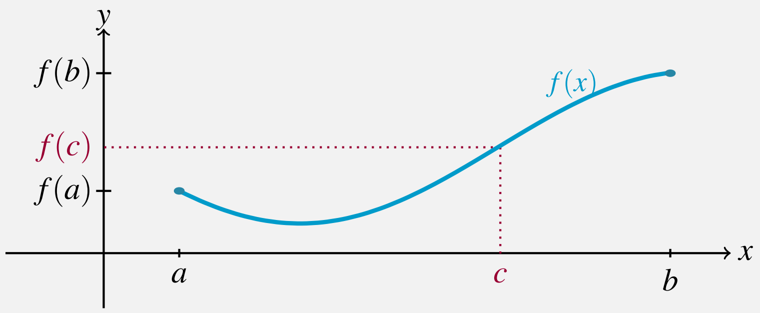
Proposition 3.3.4 — The continuous extension. Let $f : D \to \mathbb{R}$ a continuous function and $a \in \mathbb{R} \setminus D$ such that f be defined at the neighborhood of a.
If $f \xrightarrow[x \to a]{} \ell \in \mathbb{R}$ then the function $\tilde{f} : D \cup \{a\} \to \mathbb{R}$ defined by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in D \\ \ell & \text{if } x = a \end{cases}$$

is the unique extension by continuity of f in the domain $D \cup \{a\}$.

■ **Example 3.15** Let $f :]0,+\infty[\to \mathbb{R}$ defined by $f(x) = x \ln x$. f is continuous in $\mathbb{R}^{+*}$ and $\lim_{x \to 0^+} f(x) = 0$.
We extend f by continuity in 0 by putting $f(0) = 0$. ■

Theorem 3.3.5 — Intermediate value theorem. Let I an interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ and $a, b \in I$ such that $a \leqslant b$.
If f is continuous then f takes all the Intermediate values between $f(a)$ and $f(b)$.



R This theorem allows us to prove the existence of solutions for the equation $f(x) = \alpha$, but does not invoke its uniqueness which is not always true.
An injectivity argument, for example by strict monotony, allows us to obtain an eventual uniqueness of such a x.

Corollary 3.3.6 If a continuous function $f : I \to \mathbb{R}$ takes a positive and a negative values then f cancels out.

■ **Example 3.16** The equation $-3x^5 + 2x^4 - x^2 + 7x + 5 = 0$ possess at least one real solution.
The function $f :\ x \to -3x^5 + 2x^4 - x^2 + 7x + 5$ is continuous because it is polynomial.
Since $\lim_{x\to+\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = +\infty$, the function f takes a positive and negative value and so this function cancels out at least once.
Generally speaking, the above step adapts to polynomial functions of odd degrees. ■

Theorem 3.3.7 — Image of a segment. Let $a, b \in \mathbb{R}$ such that $a \leqslant b$ and $f : [a,b] \to \mathbb{R}$.
If f is continuous then f admits a minima and a maxima. We say that f is bounded and reaches its boundaries.
Thus

$$\exists c,d \in [a,b],\ \forall x \in [a,b] :\ f(c) \leqslant f(a) \leqslant f(d)$$

We can write $f(c) = \min_{[a,b]} f$ and $f(d) = \max_{[a,b]} f$, and so $f([a,b]) = \left[\min_{[a,b]} f,\ \max_{[a,b]} f\right]$.

■ **Example 3.17** Let $f : x \to x^2$, we have : $f([-2,3]) = [0,9]$. ■

Theorem 3.3.8 — The monotone bijection and the reciprocal mapping. If $f : I \to \mathbb{R}$ is a continuous, strictly monotone function defined in an interval I then

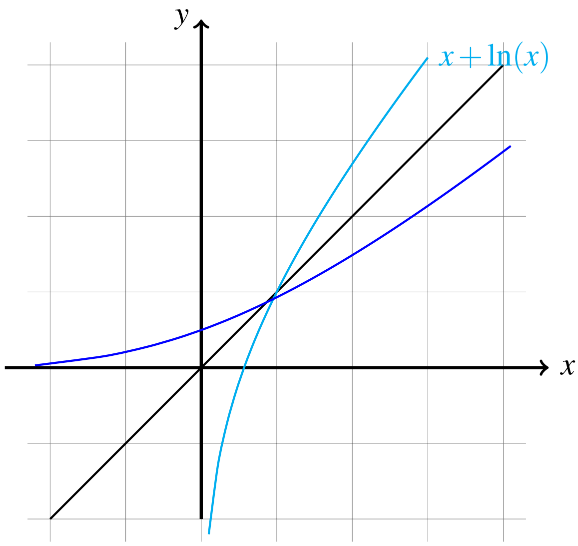
- $f(I)$ is an interval of the same type as I and whose extremities are the limits of f at the ends of I.
- f performs a bijection of I to $f(I)$.
- Its reciprocal mapping $f^{-1} : f(I) \to I$ is continuous, of the same monotony as f and limits of f^{-1} at the ends of $f(I)$ are extremities of I.

R By this theorem, it is possible to deduce the bijections from the variations table. It is this theorem that allows the definition of multiple usual function such as reciprocal mappings of knows functions. (We will use it in the elementary functions course).

■ **Example 3.18** Let $f : \mathbb{R}^{+*} \to \mathbb{R}$ defined by $f(x) = x + \ln x$.
The function f is continuous and strictly increasing.
Its limits in 0^+ and $+\infty$ are respectively $-\infty$ and $+\infty$.
By the theorem of monotone bijection, we can affirm that f performs a bijection from $\mathbb{R}^{+*}$ to $\left[\lim_{0^+} f, \lim_{+\infty} f\right] = \mathbb{R}$.

In addition, its reciprocal mapping $f^{-1}:\mathbb{R}\rightarrow\mathbb{R}^{+*}$ is continuous, strictly increasing and we have

$$\lim_{x\rightarrow-\infty} f^{-1}(x)=0^{+} \quad\text{and}\quad \lim_{x\rightarrow+\infty} f^{-1}(x)=+\infty.$$



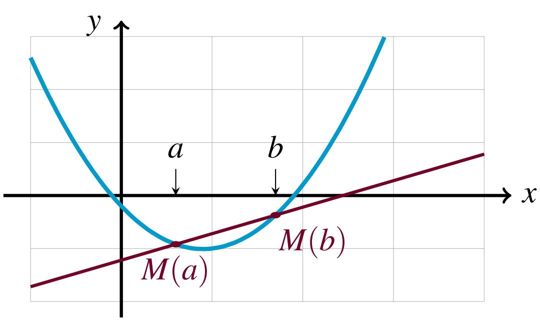
■

3.4 Differentiation

In this section, we unravel the principles and techniques behind finding derivatives, exploring their applications in understanding rates of change and the behavior of functions.
In this part, I and J denote non-singular intervals of $\mathbb{R}$ i.e. Non-empty intervals and not reduced at one point.

Definition 3.4.1 — Variation rate. Let $f:I\rightarrow\mathbb{R}$ et Γ_f its graph.
For $a,b\in I$ distinct, we call variation rate of f between a and b the real :

$$\tau(a,b)=\frac{f(b)-f(a)}{b-a}$$



R $\tau(a,b)$ is the slope (or leading coefficient) of the line $(M(a)M(b))$.

Definition 3.4.2 — Differentiation on one point. We say that f is differentiable in $a \in I$ if the variation rate of f between a and x converges when $x \to a$ (with $x \neq a$) in other words if

$$\lim_{x \to a,\ x \neq a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0,\ h \neq 0} \frac{f(a+h) - f(a)}{h}$$

exists and is finite.
This limit is then called the differentiable number of f in a and is noted $f'(a)$.

Proposition 3.4.1 — Tangent. If f is differentiable in $a \in I$ then the curve Γ_f admits a tangent on a point $M(a)$ that is the line of equation T.

$$y = f'(a)(x-a) + f(a)$$

Abusively, we say that T is the tangent of f in a.

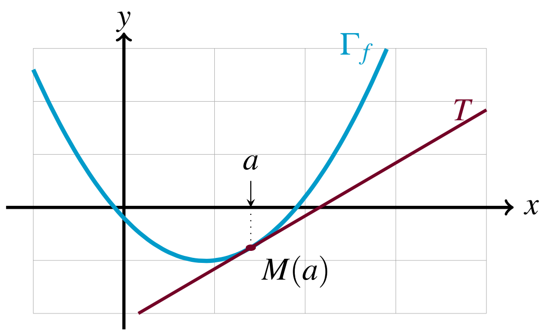
R The lines (AM) rotates around the point A and have for slope $\tau(a, a+h)$.
So when $h \to 0$, $\tau(a, a+h) \to f'(a)$, and the line (AM) extends to the line passing by A of slope $f'(a)$.

Theorem 3.4.2 — 1^{st} Order Series Expansion. If f is differentiable in $a \in I$ then, when $x \to a$

$$f(x) = f(a) + f'(a)(x-a) + o(x-a)$$

where $o(x-a) := (x-a)\varepsilon(x)$ with $\varepsilon(x) \xrightarrow[x \to a]{} 0$.
This relation is called a 1^{st} order series expansion of f in a.

R This relation means that the tangent in a is the closest line to the curve Γ_f at the neighborhood of the point of the abscissa a.



Corollary 3.4.3 If $f : I \to \mathbb{R}$ is differentiable in a then f is continuous in a.

Proof. When $x \to a$

$$f(x) = f(a) + f'(a)(x-a) + o(x-a) \to f(a)$$

■

■ **Example 3.19** Let $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=C$ a constant function.
Let $a\in\mathbb{R}$.
When $x\to a$ (with $x\neq a$)

$$\frac{f(x)-f(a)}{x-a}=0\xrightarrow[x\to a]{}0$$

Subsequently f is differentiable in $a\in\mathbb{R}$ and $f'(a)=0$. ■

■ **Example 3.20** Let $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x$
Let $a\in\mathbb{R}$.
When $x\to a$ (with $x\neq a$)

$$\frac{f(x)-f(a)}{x-a}=\frac{x-a}{x-a}=1\xrightarrow[x\to a]{}1$$

Subsequently, f is differentiable in $a\in\mathbb{R}$ and $f'(a)=1$. ■

Proposition 3.4.4 If f is continuous in $a\in I$ and if $\lim\limits_{h\to 0,\ h\neq 0}\dfrac{f(a+h)-f(a)}{h}=+\infty$ or $-\infty$ then f is not differentiable in a and Γ_f admits a vertical tangent in $M(a)$ whose line of equation $x=a$. Abusively, we say that f admits a vertical tangent in a.

■ **Example 3.21** Let $f:\mathbb{R}^+\to\mathbb{R}$ defined by $f(x)=\sqrt{x}$.
When $x\to 0^+$

$$\frac{f(x)-f(0)}{x}=\frac{1}{\sqrt{x}}\xrightarrow[x\to 0^+]{}+\infty$$

The square root function is not differentiable in 0 but its graph shows a vertical tangent in 0. ■

R There exists continuous functions which do not admit a tangent in a point such as the function f defined by

$$f(x)=\begin{cases}x\sin(1/x) & \text{if } x\neq 0\\ 0 & \text{otherwise}\end{cases}$$

Even though f is continuous on $\mathbb{R}$, it does not admit a tangent in 0.

Definition 3.4.3 — Derivative function. We say that $f:I\to\mathbb{R}$ is differentiable if f is differentiable in all points of I.
We introduce its derivative function which is the mapping f' from I to $\mathbb{R}$ whose for $x\in I$ associates a derived number of f in x.
We write $\mathscr{D}(I,\mathbb{R})$ the set of derivative (differentiable) functions from I to $\mathbb{R}$.

■ **Example 3.22** Let us remind some formulas of differentiation which should be known :
The constant functions have null derivatives in $\mathbb{R}$.
$(e^x)'=e^x$ on $\mathbb{R}$ and $(\ln x)'=\dfrac{1}{x}$ on $\mathbb{R}^{+*}$.
For $n\in\mathbb{Z}$, $(x^n)'=nx^{n-1}$ and $\left(\dfrac{1}{x^n}\right)'=-\dfrac{n}{x^{n+1}}$ on $\mathbb{R}^{+*}$ or $\mathbb{R}^{-*}$.
For $\alpha\in\mathbb{R}$, $(x^\alpha)'=\alpha x^{\alpha-1}$ and particularly on $\left(\sqrt{x}\right)'=\dfrac{1}{2\sqrt{x}}$ on $\mathbb{R}^{+*}$.

$(\cos x)' = -\sin x$, $(\sin x)' = \cos x$ on $\mathbb{R}$.

$(\tan x)' = 1 + \tan^2 x = \dfrac{1}{\cos^2 x}$ on $]-\pi/2, \pi/2[+ k\pi$ (with $k \in \mathbb{Z}$). ■

Proposition 3.4.5 If $f : I \to \mathbb{R}$ is differentiable then f is continuous.

R There exists continuous functions that are not differentiable such as the continuous functions $\sqrt{.}$ and $|\,.\,|$ both not differentiable in 0.
We point out the existence of Bolzano's function which is continuous in $[0,1]$ and not differentiable in any point of $[0,1]$. Here is its graph

Theorem 3.4.6 — Operations on derivatives. If f and $g : I \to \mathbb{R}$ are differentiable then for all $\lambda \in \mathbb{R}$ the functions λf, $f+g$, fg are also differentiable.
We then have $(\lambda f)' = \lambda f'$, $(f+g)' = f' + g'$ and $(fg)' = f'g + fg'$.
In addition if f does not cancel in I, its inverse function $\dfrac{1}{f}$ is differentiable and

$$\left(\frac{1}{f}\right)' = \frac{-f'}{f^2}.$$

Corollary 3.4.7 If $f_1, \cdots, f_n : I \to \mathbb{R}$ are differentiable then so are $f_1 + \cdots + f_n$ and $f_1 \cdots f_n$, and in addition

$$(f_1 + \cdots + f_n)' = f_1' + \cdots + f_n'$$

and

$$(f_1 \cdots f_n)' = \sum_{i=1}^{n} f_1 \cdots f_{i-1} (f_i)' f_{i+1} \cdots f_n$$

R Particularly

$$(fgh)' = f'gh + fg'h + fgh'$$

Corollary 3.4.8 Let $f, g : I \to \mathbb{R}$ with g not canceling out.
If f and g are differentiable then so is $\frac{f}{g}$ and

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$

Theorem 3.4.9 — Derivative of a composite function. Let $f : I \to \mathbb{R}$ and $\varphi : J \to \mathbb{R}$ satisfying $f(I) \subset J$.
If f and φ are differentiable then so is $\varphi \circ f$ and

$$(\varphi \circ f)' = f' \times \varphi' \circ f$$

■ **Example 3.23** By differentiation of the composite function and for a function u differentiable :

$$\left(\frac{1}{1+x^2}\right)' = \left(\frac{1}{u}\right)' = \frac{-u'}{u^2} = \frac{-2x}{(1+x^2)^2}.$$

$$(\mathrm{e}^u)' = u'\mathrm{e}^u,\ (\ln u)' = \frac{u'}{u},\ (u^\alpha)' = \alpha u' u^{\alpha-1}.$$

$$(\cos u)' = -u'\sin u,\ (\sin u)' = u'\cos u,\ (\tan u)' = u'\left(1+\tan^2 u\right).$$ ■

Theorem 3.4.10 — Reciprocal function derivative. If $f : I \to J$ is a differentiable bijection verifying $\forall x \in I, f'(x) \neq 0$ then its reciprocal mapping f^{-1} is differentiable and

$$(f^{-1})' = \frac{1}{f' \circ f^{-1}}$$

■ **Example 3.24** We can introduce the neperian logarithm function as the primitive on $\mathbb{R}^{+*}$ of the inverse function that cancels out in 1. By showing that this function performs a bijection from $\mathbb{R}^{+*}$ to $\mathbb{R}$, we can introduce its reciprocal mapping, the exponential function and the previous derivative formula gives

$$(\mathrm{e}^x)' = \frac{1}{(\ln)'(\mathrm{e}^x)} = \frac{1}{1/\mathrm{e}^x} = \mathrm{e}^x$$

■

Definition 3.4.4 — Right derivative number, left derivative number. Let $f : I \to \mathbb{R}$ and $a \in I$.
If a is not the right boundary of I, we say that f is right (r) differentiable in a if the variation rate $\frac{f(a+h)-f(a)}{h}$ converges when $h \to 0^+$. We then put

$$f_r'(a) = \lim_{h \to 0^+} \frac{f(a+h)-f(a)}{h}$$

If a is not the left boundary of I, we say that f is left (l) differentiable in a if the variation rate

$\dfrac{f(a+h)-f(a)}{h}$ converges when $h \to 0^-$. We then put

$$f_l'(a) = \lim_{h \to 0^-} \frac{f(a+h)-f(a)}{h}$$

■ **Example 3.25** For $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$, we have $f_r'(0) = 1$ and $f_l'(0) = -1$. ■

Proposition 3.4.11 Let $f : I \to \mathbb{R}$ and $a \in I$ whose not a boundary of I. We have an equivalence between :

- f is differentiable in a.
- f is left and right differentiable in a and $f_l'(a) = f_r'(a)$.

In addition, if this is the case, then $f'(a) = f_l'(a) = f_r'(a)$.

■ **Example 3.26** Let us study the differentiation of $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x^2, & \text{otherwise} \end{cases}$$

Over $]0, +\infty[$, we have $f(x) = x^2$ and so f is differentiable and $f'(x) = 2x$.
Over $]-\infty, 0[$, we have $f(x) = -x^2$ and so f is differentiable and $f'(x) = -2x$.
Differentiation in $a = 0$:
When $h \to 0^+$, $\dfrac{f(h)-f(0)}{h} = \dfrac{h^2-0}{h} \to 0$ so f is left differentiable in 0 and $f_r'(0) = 0$.
With the same method we get $f_l'(0) = 0$.
Since f is left and right differentiable in 0 with $f_l'(0) = f_r'(0)$, f is differentiable in 0 and $f'(0) = 0$. In the end f is differentiable on $\mathbb{R}$. ■

Definition 3.4.5 — Successive Derivatives. Let $f : I \to R$.
We put $f^{(0)} = f$ called derivative of order 0 of f.
If f is differentiable, we write $f^{(1)} = f'$ called derivative of order 1 of f.
If in addition, f' is differentiable, we put $f^{(2)} = f'' = (f')'$ called derivative of order 2 (or second derivative) of f.
So, closer and closer, if f possess a derivative of order $n \in \mathbb{N}$, written $f^{(n)}$ (or $D^n f$), and if it is differentiable, we put $f^{(n+1)} = \left(f^{(n)}\right)'$ called derivative of order $n+1$ of f.
We say that $f : I \to \mathbb{R}$ is n times differentiable if its derivative of order n exists.
We say that f is indefinitely differentiable if for all $n \in \mathbb{N}$, f is n times differentiable.

■ **Example 3.27** The functions $x \to e^x$, $x \to \cos x$, $x \to \sin x$ and the polynomial functions are indefinitely differentiable on $\mathbb{R}$. ■

Definition 3.4.6 — Class of a function. We say that a function $f : I \to \mathbb{R}$ is of class $\mathscr{C}^n$ (with $n \in \mathbb{N}$) if its derivative of order n exists and is continuous.
We write down $\mathscr{C}^n(I, \mathbb{R})$ the set of real functions of class $\mathscr{C}^n$ defined on I.
We say that a function $f : I \to \mathbb{R}$ is of class $\mathscr{C}^\infty$ if it is of class $\mathscr{C}^n$ for all $n \in \mathbb{N}$.
We denote $\mathscr{C}^\infty(I, \mathbb{R})$ the set of those functions.

■ **Example 3.28**

- Stating that $f : I \to \mathbb{R}$ is of class $\mathscr{C}^0$ means that f is continuous.
- Saying that $f : I \to \mathbb{R}$ is of class $\mathscr{C}^1$ signifies that f is differentiable and that its derivative is continuous.
- Let $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 \sin 1/x & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

  We can easily verify that f is differentiable on $\mathbb{R}$ with $f'(0) = 0$.
  However for $x \neq 0$, $f'(x) = 2x \sin 1/x - \cos 1/x$ diverges when $x \to 0$.
  So f' is not continuous in 0, this function is not $\mathscr{C}^1$.

■

Proposition 3.4.12 — Bernoulli's rule or L'Hôpital's rule. Let $f(x) = \dfrac{g(x)}{h(x)}$ with f and g differentiable, and we wish to calculate $\lim\limits_{x \to a} f(x)$.

1. Let us start by replacing all x by the value of a in the function $f(x)$.
2. If you obtain $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ then derive the numerator $g(x)$ and the denominator $h(x)$ to get $\dfrac{g'(x)}{h'(x)}$.
   Note that Bernoulli's rule can be summarized by the following formula: (only use in case of undetermined cases such as the following):

$$\lim_{x \to a} \frac{g(x)}{h(x)} = \lim_{x \to a} \frac{g'(x)}{h'(x)}$$

3. If you get a real number, ∞ or 0, you have your answer, the calculus is done. If you get $\dfrac{0}{0}$ again or $\dfrac{\infty}{\infty}$ repeat step 2 (many times, until you get your answer).
   Otherwise if the limit of the quotient of the derivatives does not exist, this method does not work.

■ **Example 3.29**

- $\lim\limits_{x \to 0} \dfrac{\cos 2x - 1}{x^3 + 5x^2} = \lim\limits_{x \to 0} \dfrac{-2\sin 2x}{3x^2 + 10x} = \lim\limits_{x \to 0} \dfrac{-4\cos 2x}{6x + 10} = \dfrac{-4}{10}$
- $\lim\limits_{x \to +\infty} \dfrac{\sqrt{x}}{\ln x} = \lim\limits_{x \to +\infty} \dfrac{\frac{1}{2\sqrt{x}}}{\frac{1}{x}} = \lim\limits_{x \to +\infty} \dfrac{\sqrt{x}}{2} = +\infty$
- This rule is only used in undetermined case :
  $4 = \lim\limits_{x \to 1} \dfrac{3x^2 + 1}{2x - 1} \neq \lim\limits_{x \to 1} \dfrac{6x}{2} = 3$
- We cannot use this rule to calculate $\lim\limits_{x \to 0} \dfrac{x^2 \sin(1/x)}{x}$
  because $\dfrac{2x \sin(1/x) - \cos 1/x}{1}$ does not admit a limit in 0.

■

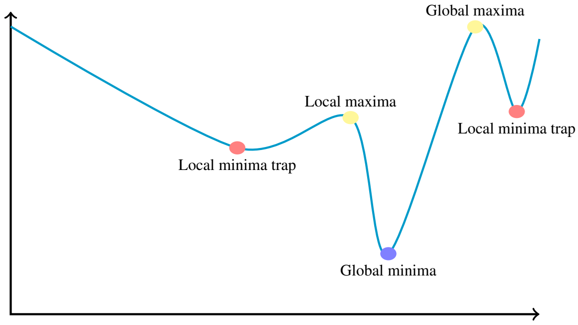
Definition 3.4.7 — Local maximum and minimum. We say that a function $f : I \to \mathbb{R}$ admits a local minimum in $a \in I$ if there exists $\alpha > 0$ verifying

$$\forall x \in I \cap [a-\alpha,\ a+\alpha],\ f(x) \geqslant f(a)$$

We say that a function $f : I \to \mathbb{R}$ admits a local maximum in $a \in I$ if there exists $\alpha > 0$ verifying

$$\forall x \in I \cap [a-\alpha,\ a+\alpha],\ f(x) \leqslant f(a)$$

In both cases we talk about local extremum in a.

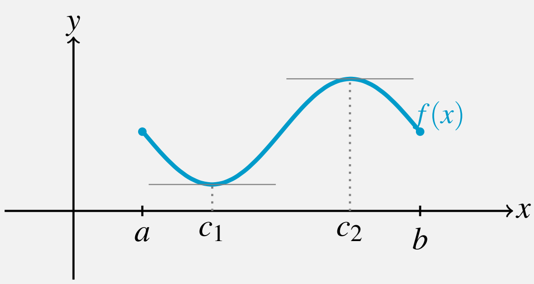


■ **Example 3.30** Extremums (global) of f are mostly local extrema. ■

Theorem 3.4.13 If $f : I \to \mathbb{R}$ admits a local extrema in $a \in I$ whose not an extrema of I and if f is differentiable in a then $f'(a) = 0$.

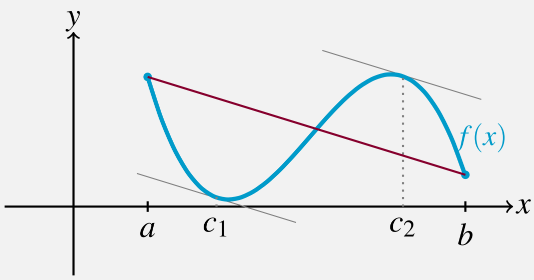
R The function $x \to x^2$ defined on $\mathbb{R}$ admits a global minimum in 0, so $f'(0) = 0$ (which can be verified).
The converse of this result is not true, even though the derivative of the function $x \to x^3$ cancels out in 0, it does not admit an extrema in 0.

Theorem 3.4.14 — Rolle's lemma or theorem. Let $a, b \in \mathbb{R}$ with $a < b$.
If $f : [a,b] \to \mathbb{R}$ is continuous over $[a,b]$, differentiable over $]a,b[$ and if $f(a) = f(b)$ then there exists $c \in]a,b[$ verifying $f'(c) = 0$.



■ **Example 3.31** Over $[-3,3]$ Roll's theorem can be used for the function $x \to x^2$ because it satisfies the conditions of the theorem.
For the function $x \to |x|$ we cannot apply it because $x \to |x|$ is not differentiable in 0 so it is not differentiable over the interval $]-3,3[$. ■

Theorem 3.4.15 — Mean value theorem. Let $a, b \in \mathbb{R}$ with $a < b$.
If $f : [a,b] \to \mathbb{R}$ is continuous over $[a,b]$, differentiable over $]a,b[$ then there exists $c \in]a,b[$ verifying $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.



■ **Example 3.32** Let us use the mean value theorem for the function $x \to f(x) = x^2 + 2x$ over $[0,1]$.
We have f is continuous over $[0,1]$, differentiable over $]0,1[$, so there exists $c \in]0,1[$ such that :
$\dfrac{f(1)-f(0)}{1-0} = f'(c)$ i.e. $f'(c) = 3$ i.e. $2c+2 = 3$ i.e. $c = 1/2$. ■

Theorem 3.4.16 — Variations of derivative function. Let $f : I \to \mathbb{R}$ a differentiable function.
The function f is increasing (resp. strictly increasing) if, and only if

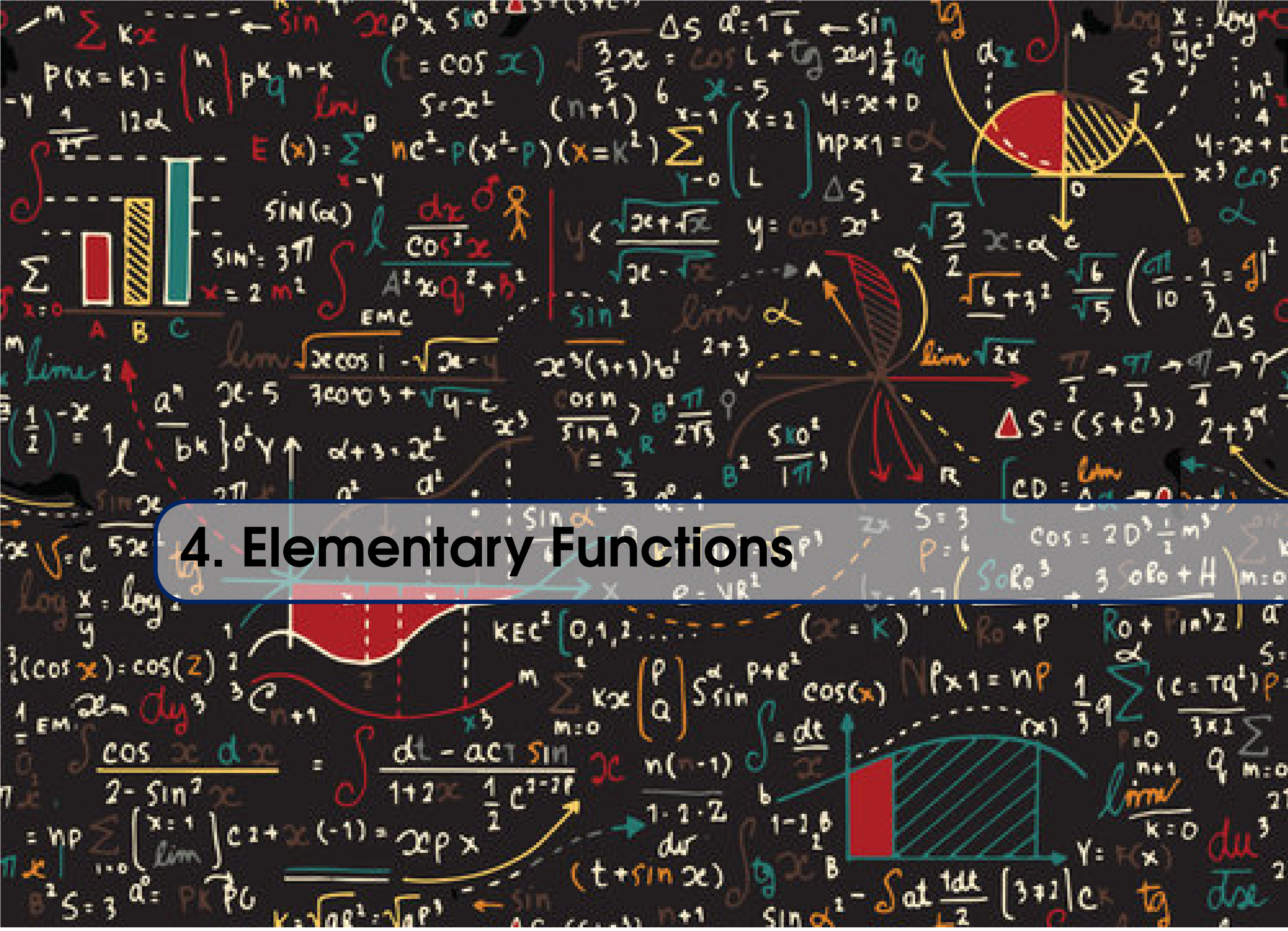
$$\forall x \in I,\ f'(x) \geqslant 0 \text{ (resp. } f'(x) > 0)$$

The function f is decreasing (resp. strictly decreasing) if, and only if

$$\forall x \in I,\ f'(x) \leqslant 0 \text{ (resp. } f'(x) < 0)$$

The function f is constant if, and only if

$$\forall x \in I,\ f'(x) = 0.$$

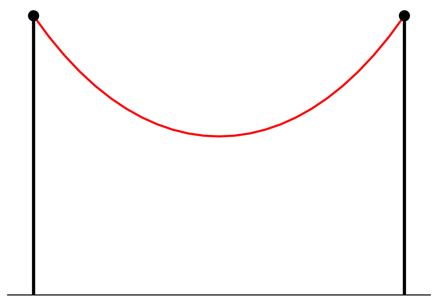


4. Elementary Functions

This chapter explores the properties and behaviors of elementary functions, laying the groundwork for a deeper understanding of mathematical analysis and problem-solving.

4.1 Motivation

We already know some classic functions : exp, ln, cos, sin, tan. In this chapter, we are going to add some new functions to our catalog : cosh, sinh, tanh, arccos, arcsin, arctan, arg cosh, arg sinh, arg tanh.



For example when a necklace is held between two hands then the drawn curve is a **chain** of which the equation involves the hyperbolic cosine and a parameter a (that depends on the length of the wire and the spacing of the posts)

$$y = a\cosh\left(\frac{x}{a}\right).$$

4.2 Logarithm and exponential

In this section, we unravel the properties, relationships, and unique characteristics of logarithmic and exponential functions.

Definition 4.2.1 — Logarithm. There exists a unique fucntion, written $\ln :]0,+\infty[\to \mathbb{R}$ such that :

$$\ln'(x) = \frac{1}{x} \quad \text{(for all } x > 0) \qquad \text{and} \qquad \ln(1) = 0.$$

In addition, this function verifies (for all $a, b > 0$) :

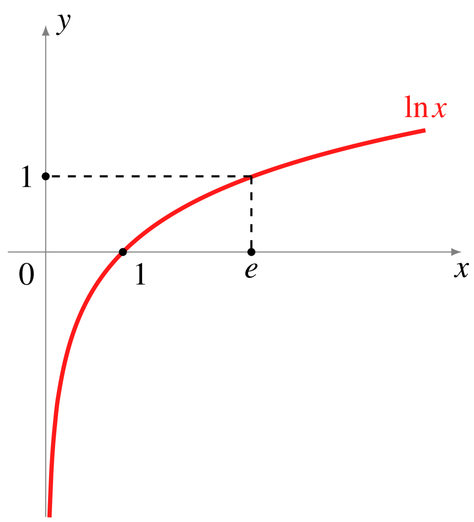
1. $\ln(a \times b) = \ln a + \ln b$,
2. $\ln\left(\frac{1}{a}\right) = -\ln a$,
3. $\ln(a^n) = n\ln a$, (for all $n \in \mathbb{N}$)
4. ln is a continuous function, strictly increasing and defines a bijection of $]0,+\infty[$ on $\mathbb{R}$,
5. $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$,
6. We have $\ln x \leq x - 1$ (for all $x > 0$).

R $\ln x$ is called the **Natural Logarithm**. It is caracterized by $\ln(e) = 1$. We define the **base** a**'s logarithm** by

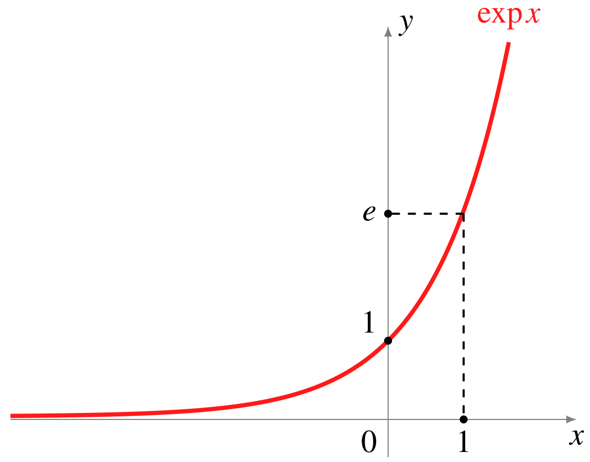
$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$

So that $\log_a(a) = 1$.

For $a = 10$ we get the **decimal logarithm** $\log_{10}$ such that $\log_{10}(10) = 1$ (and so $\log_{10}(10^n) = n$).



Definition 4.2.2 — Exponential. The reciprocal bijection of $\ln :]0,+\infty[\to \mathbb{R}$ is called the **exponential** function, written $\exp : \mathbb{R} \to]0,+\infty[$.



For $x \in \mathbb{R}$ we also write down e^x for $\exp(x)$.

Proposition 4.2.1 The exponential function verifies the following properties :

- $\boxed{\exp(\ln x) = x \text{ for all } x > 0}$ and $\boxed{\ln(\exp x) = x \text{ for all } x \in \mathbb{R}}$
- $\exp(a+b) = \exp(a) \times \exp(b)$
- $\exp(nx) = (\exp x)^n$
- $\exp : \mathbb{R} \to]0,+\infty[$ is a continuous function, strictly increasing verifying $\lim_{x \to -\infty} \exp x = 0$ and $\lim_{x \to +\infty} \exp = +\infty$.
- The exponential function is a a differentiable function and $\exp' x = \exp x$, for all $x \in \mathbb{R}$. In addition $\exp x \geq 1 + x$.

4.3 Power and Comparison

This section delves into the intricacies of power functions and their role in mathematical analysis, providing insights into their growth rates and relationships with other functions.

Definition 4.3.1 By definition, for $a > 0$ and $b \in \mathbb{R}$,

$$\boxed{a^b = \exp\left(b \ln a\right)}$$

R

- $\sqrt{a} = a^{\frac{1}{2}} = \exp\left(\frac{1}{2} \ln a\right)$.
- $\sqrt[n]{a} = a^{\frac{1}{n}} = \exp\left(\frac{1}{n} \ln a\right)$ (n^{th} **root**).
- We also write down $\exp x$ by e^x which is justified by the following calculus : $e^x = \exp(x \ln e) = \exp(x)$.
- The functions $x \mapsto a^x$ are also called exponential functions and systematically reduce to the classical exponential function by the equality $a^x = \exp(x \ln a)$. Do not confuse this functions with the power functions $x \mapsto x^a$.

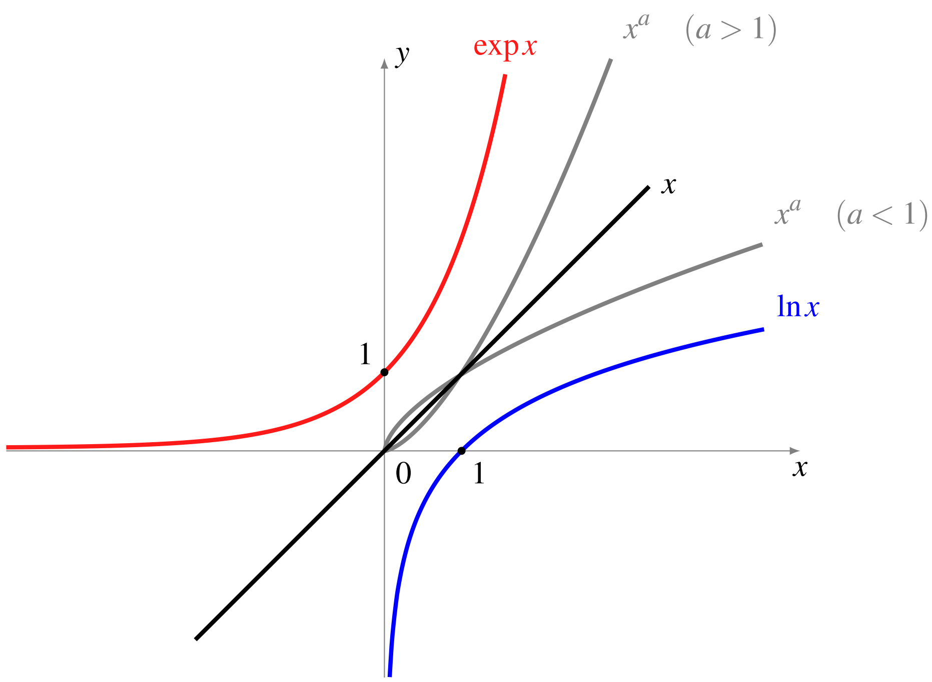
Proposition 4.3.1 — Properties. Let $x, y > 0$ and $a, b \in \mathbb{R}$.

- $\boxed{x^{a+b} = x^a x^b}$
- $\boxed{x^{-a} = \frac{1}{x^a}}$
- $\boxed{(xy)^a = x^a y^a}$
- $\boxed{(x^a)^b = x^{ab}}$
- $\boxed{\ln(x^a) = a \ln x}$

Let us compare the functions $\ln x$, $\exp x$ with x :

Proposition 4.3.2

$$\lim_{x\to+\infty} \frac{\ln x}{x} = 0 \qquad \text{and} \qquad \lim_{x\to+\infty} \frac{\exp x}{x} = +\infty.$$



4.4 Inverse Circular Functions

In this section, we delve into the properties and applications of inverse circular functions, unlocking the connections between angles and real numbers.

4.4.1 Arccosinus

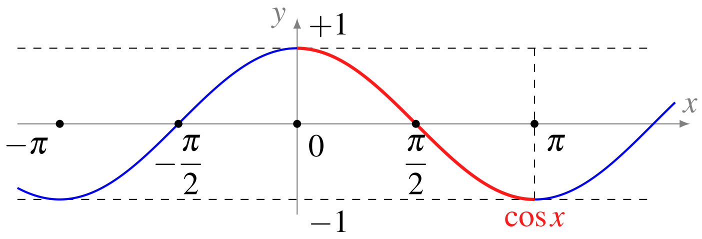
Let us consider the cosine function $\cos : \mathbb{R} \to [-1,1]$, $x \mapsto \cos x$. To obtain a bijection from this function, we must consider restricting the cosine to the interval $[0,\pi]$. In this interval, the cosinus function is continuous and stricly decreasing, so the restriction

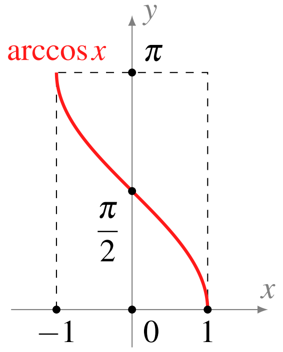
$$\cos_| : [0,\pi] \to [-1,1]$$

is a bijection.

Definition 4.4.1 — Arccosinus. The reciprocal bijection of the function cos is the **arccosinus** function :

$$\arccos : [-1,1] \to [0,\pi]$$





Proposition 4.4.1 — Properties. We have by definition the reciprocal bijection :

$$\cos\big(\arccos(x)\big) = x \quad \forall x \in [-1,1]$$
$$\arccos\big(\cos(x)\big) = x \quad \forall x \in [0,\pi]$$

In other words :

$$\text{If} \quad x \in [0,\pi] \qquad \cos(x) = y \iff x = \arccos y$$

Let us finish with the derivative function of arccos:

$$\arccos'(x) = \frac{-1}{\sqrt{1-x^2}} \qquad \forall x \in]-1,1[$$

4.4.2 Arcsinus

Definition 4.4.2 — Arcsinus. The restriction

$$\sin_| : [-\tfrac{\pi}{2}, +\tfrac{\pi}{2}] \to [-1,1]$$

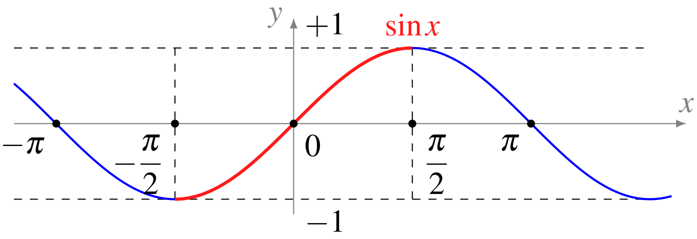
is a bijection. Its reciprocal bijection is the function **arcsinus** :

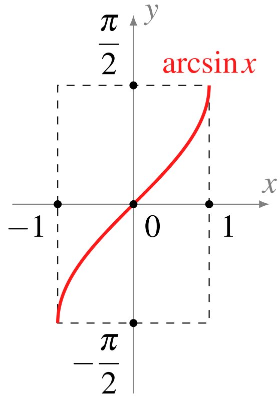
$$\arcsin : [-1,1] \to [-\tfrac{\pi}{2}, +\tfrac{\pi}{2}].$$

$$\sin\big(\arcsin(x)\big) = x \quad \forall x \in [-1,1]$$
$$\arcsin\big(\sin(x)\big) = x \quad \forall x \in [-\frac{\pi}{2}, +\frac{\pi}{2}]$$

$$\text{If} \quad x \in [-\frac{\pi}{2}, +\frac{\pi}{2}] \qquad \sin(x) = y \iff x = \arcsin y$$

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}} \qquad \forall x \in]-1, 1[$$





4.4.3 Arctangent

Definition 4.4.3 — Arctangent. The restriction

$$\tan_| :]-\tfrac{\pi}{2}, +\tfrac{\pi}{2}[\to \mathbb{R}$$

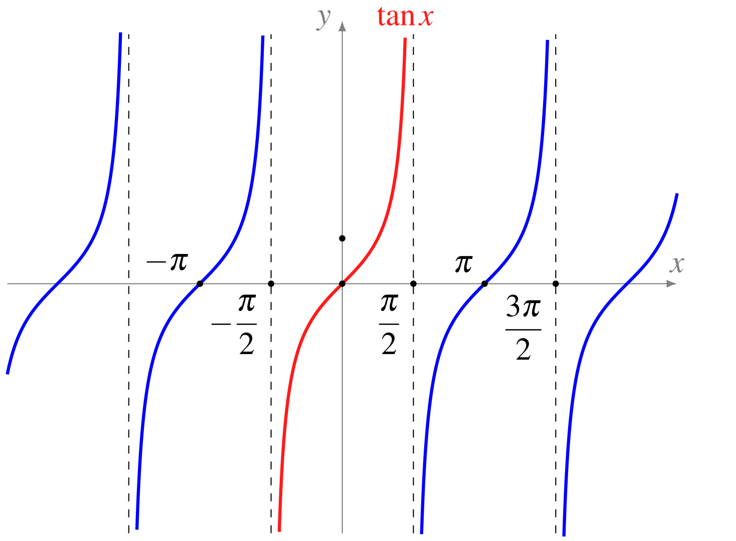
is a bijection. Its reciprocal bijection is the function **arctangente** :

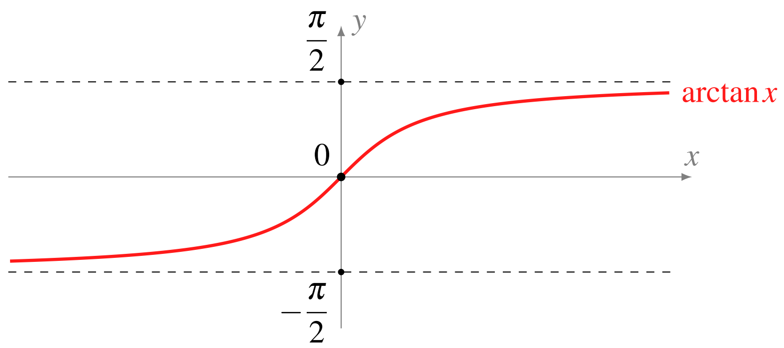
$$\arctan : \mathbb{R} \to]-\tfrac{\pi}{2}, +\tfrac{\pi}{2}[.$$

$$\tan\big(\arctan(x)\big) = x \quad \forall x \in \mathbb{R}$$
$$\arctan\big(\tan(x)\big) = x \quad \forall x \in]-\frac{\pi}{2}, +\frac{\pi}{2}[$$

$$\text{If} \quad x \in]-\frac{\pi}{2}, +\frac{\pi}{2}[\qquad \tan(x) = y \iff x = \arctan y$$

$$\arctan'(x) = \frac{1}{1+x^2} \qquad \forall x \in \mathbb{R}$$





4.5 Hyperbolic Functions and inverses

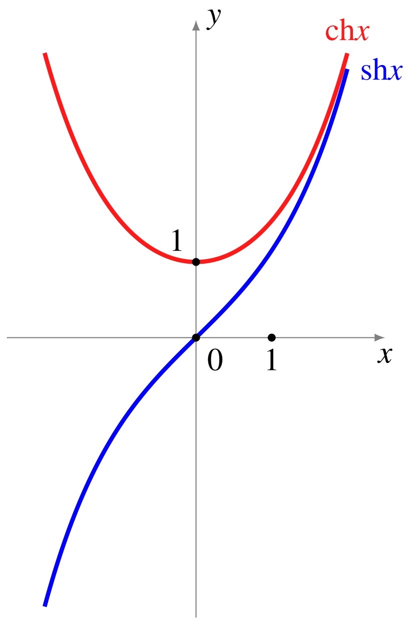
In this section, we unravel the properties, relationships, and applications of hyperbolic functions, as well as their inverse counterparts.

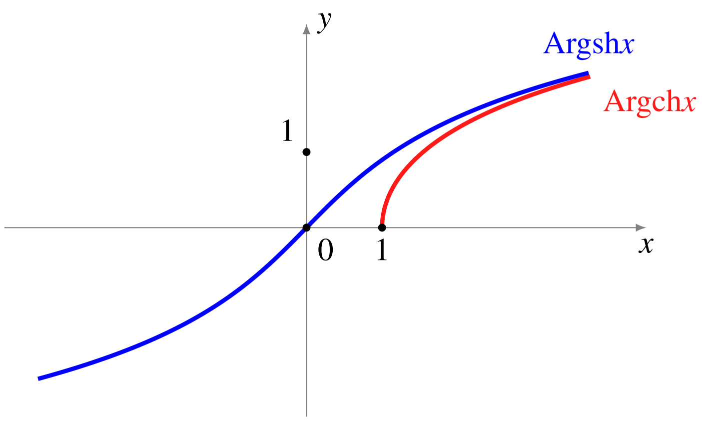
4.5.1 Hyperbolic Cosinus and its inverse

Definition 4.5.1 — Hyperbolic Cosinus. For $x \in \mathbb{R}$, The **hyperbolic cosinus** is :

$$\boxed{\cosh x = \frac{e^x + e^{-x}}{2}}$$

The restriction $\cosh_| : [0, +\infty[\to [1, +\infty[$ is a bijection. Its reciprocal bijection is the function $\arg\cosh : [1, +\infty[\to [0, +\infty[$.





4.5.2 Hyperbolic Sinus and its inverse

Definition 4.5.2 — Hyperbolic Sinus and its inverse. For $x \in \mathbb{R}$, the **hyperbolic sinus** is :

$$\boxed{\sinh x = \frac{e^x - e^{-x}}{2}}$$

$\sinh : \mathbb{R} \to \mathbb{R}$ Is a continuous function, differentiable, strictly increasing and verifying $\lim_{x\to-\infty} \sinh x = -\infty$ and $\lim_{x\to+\infty} \sinh x = +\infty$, so it's a bijection. Its reciprocal bijection is the function $\arg\sinh : \mathbb{R} \to \mathbb{R}$.

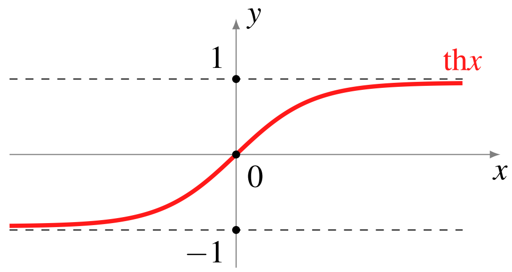
Proposition 4.5.1
- $\cosh^2 x - \sinh^2 x = 1$
- $\cosh' x = \sinh x$, $\sinh' x = \cosh x$
- $\arg\sinh : \mathbb{R} \to \mathbb{R}$ is strictly increasing and continuous.
- $\arg\sinh$ is a differentiable function and $\arg\sinh' x = \dfrac{1}{\sqrt{x^2+1}}$.
- $\arg\sinh x = \ln\left(x + \sqrt{x^2+1}\right)$.

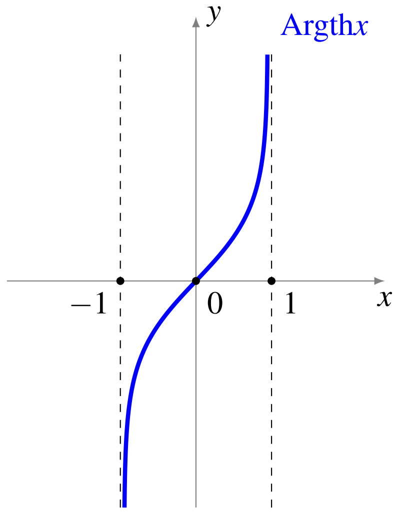
4.5.3 Hyperbolic Tangent and its inverse

Definition 4.5.3 — Hyperbolic Tangent. By definition the **hyperbolic tangent** is :

$$\boxed{\tanh x = \frac{\sinh x}{\cosh x}}$$

The function $\tanh : \mathbb{R} \to]-1,1[$ is a bijection, we write $\arg\tanh :]-1,1[\to \mathbb{R}$ its reciprocal bijection.





4.5.4 Hyperbolic Trigonometry

Proposition 4.5.2 — Hyperbolic Trigonometry.

$$\begin{aligned}
1 &= \cosh^2 x - \sinh^2 x \\
\cosh(a+b) &= \cosh a \cdot \cosh b + \sinh a \cdot \sinh b \\
\sinh(a+b) &= \sinh a \cdot \cosh b + \sinh b \cdot \cosh a \\
\cosh' x &= \sinh x \qquad \sinh' x = \cosh x \\
\tanh' x &= 1 - \tanh^2 x = \frac{1}{\cosh^2 x} \\
\arg\cosh' x &= \frac{1}{\sqrt{x^2-1}} \quad (x > 1) \qquad \arg\sinh' x = \frac{1}{\sqrt{x^2+1}} \\
\arg\tanh' x &= \frac{1}{1-x^2} \quad (|x| < 1) \\
\arg\cosh x &= \ln\left(x + \sqrt{x^2-1}\right) \quad (x \geq 1) \\
\arg\sinh x &= \ln\left(x + \sqrt{x^2+1}\right) \quad (x \in \mathbb{R}) \\
\arg\tanh x &= \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \quad (-1 < x < 1)
\end{aligned}$$

5. Series Exapansion

5.1 Introduction and Prerequisites

In physics and mathematics, a series expansion (denoted SE) of a function at a point is a polynomial approximation of this function at the neighborhood of this point, i.e. the writing of this function in the form of the sum of :

- a polynomial function denoted by $P_n(x)$ and
- a negligible remainder noted $R_n(x)$ in the neighborhood of the considered point.

In physics, the expansion series make it possible to approach the functions to simplify the calculations.
In mathematics, they make it easier to find limits of functions, to calculate derivatives, to prove that a function is integrable or not, or to study the positions of curves in relation to tangents.

Before studying the expansion series strictly speaking, it is necessary to make some reminders on the small o of x^n, noted $o(x^n)$.
We consider two functions f and g, and a real number a, such that : $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = 0$.

Then we can say that f is negligible compared to g in the neighborhood of a (it is important

to specify in the neighborhood of which point!).
Indeed, in the neighborhood of 0 for example, x is negligible compared to $1/x$, but in the neighborhood of $+\infty$, it is $1/x$ that is negligible compared to x. So just saying $1/x$ is negligible compared to ... doesn't make sense if you don't specify near which point.
When f is negligible compared to g in the neighborhood of a, we then write : $f(x) = o(g(x))$.

■ **Example 5.1** In the neighborhood of 0 : $x^6 = o(x^3)$, $x^2 = o(x)$.

$$\forall p > n,\ x^p = o(x^n).$$

■

5.2 Taylor's theorem

Definition 5.2.1 — Taylor's formula with Peano form of the remainder.. Let I be an interval of $\mathbb{R}$, a an element of I and $f : I \to \mathbb{R}$ a differentiable function at a up to a certain order $n \geqslant 1$.
Then for any real number x belonging to I, we have the Taylo's formula with Peano form of the remainder

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + o\left((x-a)^n\right)$$

or equivalently

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x)$$

where the remainder $R_n(x)$ is a negligible function with respect to $(x-a)^n$ in the neighborhood of a.

R

- By setting $h = x - a$, this formula can also be expressed as:

$$f(a+h) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} h^k + R_n(h)$$

  where the remainder $R_n(h)$ is a negligible function compared to h^n in the neighborhood of 0.
- According to the hypotheses on the function f, we can give expressions and estimates of the remainder $R_n(x)$ to obtain other more precise formulas such as: the Taylor-Lagrange formula, Taylor-Cauchy formula and Taylor with integral remainder.

■ **Example 5.2** Consider the function $f : x \to e^x$.
f is of class $\mathscr{C}^\infty$ on $\mathbb{R}$ with $f^{(k)} = e^x$ and therefore $f^{(k)}(0) = 1$, $\forall k \in \mathbb{N}$.

By the Taylor-Young formula in the neighborhood of 0, we get

$$\begin{aligned} e^x &= \sum_{k=0}^{n} \frac{1}{k!}x^k + o(x^n) \\ &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + o(x^n) \end{aligned}$$

in particular

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + o(x^4)$$

■

R The Taylor polynomial of order 4 at the point 0 is very close to e^x in neighborhood of 0. To verify, let $P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$ and evaluate at $x = 0.1$ (close to 0), we get

$$e^{0.1} = 1.10517091$$

$$P_4(0.1) = 1.10517083$$

5.3 Series expansion

I denotes a non-singular interval and n a natural number.
The functions considered here are real-valued.

Definition 5.3.1 — Series expansion. Let a be a point from I or a finite end of I and $D = I$ or $D = I \backslash \{a\}$.
We say that $f : D \to \mathbb{R}$ admits a series expansion to the order n at a (abbreviated $SE_n(a)$) if there exist $a_0, a_1, \cdots, a_n$ such that when $x \to a$

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + o\left((x-a)^n\right)$$

The polynomial function $x \to a_0 + a_1(x-a) + \cdots + a_n(x-a)^n$ is then called **regular part** of $SE_n(a)$ of f.

R In the regular part of SE, each term of the sum is negligible compared to the one before it. A $SE_n(a)$ gives information on the behavior of f at a and only at a.

■ **Example 5.3** $SE_n(0)$ of $x \to \dfrac{1}{1-x}$
We have

$$1 + x + \cdots + x^n = \frac{1 - x^{n+1}}{1-x}$$

then

$$\frac{1}{1-x} = 1 + x + \cdots + x^n + \frac{x^{n+1}}{1-x}$$

but

$$\frac{x^{n+1}}{1-x} = x^n \frac{x}{1-x} = o(x^n)$$

since $\frac{x}{1-x} \underset{x \to 0}{\longrightarrow} 0$.

so

$$\frac{1}{1-x} = 1 + x + \cdots + x^n + o(x^n)$$

which is a series expansion to the order n at 0. ■

Proposition 5.3.1 If $f : D \to \mathbb{R}$ admits a $SE_n(a)$ of the form :

$$f(x) = a_0 + a_1(x-a) + \cdots + a_n(x-a)^n + o((x-a)^n)$$

then, for every $m \leqslant n$, f admits a $SE_m(a)$ being obtained by truncation:

$$f(x) = a_0 + a_1(x-a) + \cdots + a_m(x-a)^m + o((x-a)^m)$$

■ **Example 5.4** Let $P(x) = a_0 + a_1 x + \cdots + a_p x^p$ a polynomial function. Its $SE_n(0)$ is given by :
for $n \leqslant p$, we have $P(x) = a_0 + a_1 x + \cdots + a_n x^n + o(x^n)$.
for $n > p$, we have $P(x) = a_0 + a_1 x + \cdots + a_p x^p + o(x^p)$. ■

Theorem 5.3.2 — Uniqueness. If $f : D \to \mathbb{R}$ admits a $SE_n(a)$ then this one is unique.

Theorem 5.3.3 — Existence of SE. f admits a SE of order 0 at a if and only if f converges at a.
Moreover, if this is the case i.e. $\lim_{x \to a} f(x) = a_0$, then the SE of order 0 of f is given by $f(x) = a_0 + o(1)$.

Proposition 5.3.4 f is continuous at a and admits a SE of order 1 at a if and only if f is derivable at a.
Moreover, $f'(a) = a_1$, so the SE of order 1 of f is given by

$$f(x) = f(a) + f'(a)(x-a) + o(x-a)$$

■ **Example 5.5**

- $f(x) = \ln(x-2)$, $a = 2$.
  we have $\lim_{x \to 2^+} \ln(x-2) = -\infty$ so f does not admit a SE at $a = 2$.
- $f(x) = |x|$ converges at 0 so it admits a SE of order 0 at 0.
  On the other hand f is not differentiable at 0, so it does not admit a SE of order 1 at 0.
- $f(x) = \sqrt{1 + \sin x}$, $a = 0$.
  Since f is of class C^∞ in neighborhood of 0, it admits a Taylor expansion at 0 of order n, $\forall n \in \mathbb{N}$, therefore it admits a SE at 0 of order n, $\forall n \in \mathbb{N}$.
- $x \to \sin\left(\frac{1}{x}\right)$ does not converge at 0, so there is no SE at 0.

■

Theorem 5.3.5 — Relationship between SE and Taylor's expansion. Let $f : I \to \mathbb{R}$ and $a \in I$.
If f is of class $\mathscr{C}^n$ then f admits a series expansion of order n at a of the form :

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + o\left((x-a)^n\right)$$

R This theorem provides a sufficient condition for the existence of a $SE_n(a)$.
In other words: If f is of class $\mathscr{C}^n$ on I then it admits a $SE_n(a)$ and its polynomial part coincides with the Taylor polynomial of f of order n at a.
The converse implication is false; there exist functions admitting a $SE_n(a)$ which are not even of class $\mathscr{C}^1$ and which therefore do not admit a Taylor expansion of order n at a.
For example the function f defined by

$$f(x) = \begin{cases} 1+x+x^2+x^3\sin(1/x^2), & \text{if } x \neq 0 \\ 1, \quad \text{if } x = 0 \end{cases}$$

This function admits a $SE_2(0)$ (just notice that $x^3\sin(1/x^2) = o(x^2)$), on the other hand it is not even twice differentiable at 0 (because its first derivative is not continuous at 0).

SE at 0 of usual functions

- $\dfrac{1}{1-x} = 1+x+x^2+\cdots+x^n+o(x^n) = \displaystyle\sum_{k=0}^{n} x^k + o(x^n)$
- $\dfrac{1}{1+x} = 1-x+x^2-x^3+\cdots+(-1)^n x^n + o(x^n) = \displaystyle\sum_{k=0}^{n} (-1)^k x^k + o(x^n)$
- $e^x = 1+x+\dfrac{1}{2!}x^2+\dfrac{1}{3!}x^3+\cdots+\dfrac{1}{n!}x^n+o(x^n) = \displaystyle\sum_{k=0}^{n} \frac{1}{k!}x^k + o(x^n)$
- $\cos x = 1-\dfrac{1}{2!}x^2+\dfrac{1}{4!}x^4+\cdots+\dfrac{(-1)^n}{(2n)!}x^{2n}+o(x^{2n}) = \displaystyle\sum_{k=0}^{n} \frac{(-1)^k}{(2k)!}x^{2k}+o(x^{2n})$
- $\sin x = x-\dfrac{1}{3!}x^3+\dfrac{1}{5!}x^5+\cdots+\dfrac{(-1)^n}{(2n+1)!}x^{2n+1}+o(x^{2n+1})$
  $= \displaystyle\sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}x^{2k+1}+o(x^{2n+1})$.
- Since $\mathrm{ch}(x) = \dfrac{1}{2}\left(e^x+e^{-x}\right)$, we get by summing the expansions of e^x and of e^{-x} :
  $\mathrm{ch}(x) = 1+\dfrac{1}{2!}x^2+\dfrac{1}{4!}x^4+\cdots+\dfrac{1}{(2n)!}x^{2n}+o(x^{2n}) = \displaystyle\sum_{k=0}^{n} \frac{1}{(2k)!}x^{2k}+o(x^{2n})$
- $\mathrm{sh}(x) = x+\dfrac{1}{3!}x^3+\dfrac{1}{5!}x^5+\cdots+\dfrac{1}{(2n+1)!}x^{2n+1}+o(x^{2n+1})$
  $= \displaystyle\sum_{k=0}^{n} \frac{1}{(2k+1)!}x^{2k+1}+o(x^{2n+1})$
- $\ln(1+x) = x-\dfrac{1}{2}x^2+\dfrac{1}{3}x^3+\cdots+\dfrac{(-1)^{n-1}}{n}x^n+o(x^n) = \displaystyle\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}x^k+o(x^n)$

- For $\alpha \in \mathbb{R}$ fixed
  $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + o(x^n)$
- For $\alpha = p \in \mathbb{N}$
  $(1+x)^\alpha = \binom{p}{0} + \binom{p}{1}x + \binom{p}{2}x^2 + \cdots + \binom{p}{n}x^n + o(x^n) = \sum_{k=0}^{n}\binom{p}{k}x^k + o(x^n)$

5.4 Determination of series expansion

5.4.1 Transfering the problem to 0

Proposition 5.4.1 — Transfering the problem to 0. To determine a series expansion at a of a function $x \to f(x)$, we relocate the problem to 0 using the change of variable $x = a + h$.
We then determine a series expansion at 0 of the function $h \to f(a+h)$ then we transpose this expansion at a by replacing h by $x - a$.

■ Example 5.6

- $SE_2(1)$ of $x \to \mathrm{e}^x$.
  When $x \to 1$, we put $x = 1 + h$, $h = x - 1$ with $h \to 0$

$$\mathrm{e}^x = \mathrm{e}^{1+h} = \mathrm{e}.\mathrm{e}^h = \mathrm{e}\left(1 + h + \frac{1}{2}h^2 + o(h^2)\right) = \mathrm{e} + \mathrm{e}.h + \frac{e}{2}h^2 + o(h^2)$$

  then

$$\mathrm{e}^x = \mathrm{e} + \mathrm{e}(x-1) + \frac{\mathrm{e}}{2}(x-1)^2 + o\left((x-1)^2\right)$$

> **R** Do not develop this expression because otherwise we lose the visualization of the orders of magnitude in the neighborhood of 1.

- $SE_3(\pi/3)$ of $x \to \cos x$.
  When $x \to \pi/3$, we put $x = \pi/3 + h$, $h = x - \pi/3$ with $h \to 0$.

$$\cos x = \cos\left(\frac{\pi}{3} + h\right) = \frac{1}{2}\cos h - \frac{\sqrt{3}}{2}\sin h$$

  But $\cos h = 1 - \frac{1}{2}h^2 + o(h^3)$ and $\sin h = h - \frac{1}{6}h^3 + o(h^3)$ so

$$\cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}h - \frac{1}{4}h^2 + \frac{\sqrt{3}}{12}h^3 + o(h^3)$$

  then

$$\cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{3}\right)^3 + o\left(\left(x - \frac{\pi}{3}\right)^3\right)$$

- $SE_2(1)$ of $x \to \ln x$. When $x \to 1$, let's put $x = 1 + h$ with $h \to 0$

$$\ln x = \ln(1+h) = h - \frac{1}{2}h^2 + o(h^2) = (x-1) - \frac{1}{2}(x-1)^2 + o\left((x-1)^2\right)$$

- $DL_2(2)$ of $\sqrt{x}$
  When $x \to 2$, let's put $x = 2+h$ with $h \to 0$

$$\sqrt{x} = \sqrt{2+h} = \sqrt{2}\sqrt{1+h/2}$$

Or $\sqrt{1+u} = 1+\frac{1}{2}u - \frac{1}{8}u^2 + o(u^2)$, when $u \to 0$ then for $u = \frac{h}{2} \to 0$ we have $\sqrt{1+h/2} = 1+\frac{1}{4}h - \frac{1}{32}h^2 + o(h^2)$
then

$$\sqrt{x} = \sqrt{2} + \frac{\sqrt{2}}{4}(x-2) - \frac{\sqrt{2}}{32}(x-2)^2 + o\left((x-2)^2\right)$$

R Here the change of variable $x = 1+h$ would have been unsuitable, since when $x \to 2$, we have $h \to 1$ and not $h \to 0$.

■

5.4.2 *SE* of a product

Proposition 5.4.2 — *SE* of a product. Assuming that in neighborhood of 0 we have
$f(x) = a_0 + a_1 x + \cdots + a_n x^n + o(x^n)$ and $g(x) = b_0 + b_1 x + \cdots + b_n x^n + o(x^n)$.
The series expansion of a product is the product of the series expansions of the factors i.e.

$$f(x)g(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0)x + \cdots + (a_0 b_n + \cdots + a_n b_0)x^n + o(x^n)$$

which determines the $SE_n(0)$ of $x \to f(x)g(x)$.

■ **Example 5.7**

- $SE_3(0)$ of $x \to \frac{e^x}{1-x}$, when $x \to 0$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + o(x^3) \text{ and } \frac{1}{1-x} = 1 + x + x^2 + x^3 + o(x^3)$$

then

$$\frac{e^x}{1-x} = 1 + 2x + \frac{5}{2}x^2 + \frac{8}{3}x^3 + o(x^3)$$

- $SE_4(0)$ of $x \to \cos x.\text{ch}x$, when $x \to 0$

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4) \text{ and } \text{ch}x = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)$$

then

$$\begin{aligned}\cos x.\text{ch}x &= 1 + \left(\frac{1}{2} - \frac{1}{2}\right)x^2 + \left(\frac{1}{24} + \frac{1}{24} - \frac{1}{4}\right)x^4 + o(x^4)\\ &= 1 - \frac{1}{6}x^4 + o(x^4)\end{aligned}$$

- $SE_3(0)$ of $x \to \ln(1+x)e^x$.

Since the expansion of $\ln(1+x)$ starts by the term x, an expansion of order 2 of e^x is enough to carry out the calculations.
Indeed, by multiplying by $\ln(1+x)$, the term $o(x^2)$ of the expansion of e^x becomes $o(x^3)$.
When $x \to 0$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^3) \quad \text{and} \quad e^x = 1 + x + \frac{1}{2}x^2 + o(x^2)$$

then

$$\begin{aligned}\ln(1+x)e^x &= x + \left(1 - \frac{1}{2}\right)x^2 + \left(\frac{1}{2} - \frac{1}{2} + \frac{1}{3}\right)x^3 + o(x^3) \\ &= x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^3)\end{aligned}$$

- $SE_4(0)$ of $x \to \ln(1+x)(1-\cos x)$.
  The expansion of $\ln(1+x)$ starting by x, an expansion of order 3 of $(1-\cos x)$ is sufficient to carry out the calculations.

$$1 - \cos x = \frac{1}{2}x^2 + o(x^3)$$

  Also, the expansion of $1-\cos x$ starting by a term x^2, an expansion to the order 2 of $\ln(1+x)$ is enough.

$$\ln(1+x) = x - \frac{1}{2}x^2 + o(x^2)$$

  then

$$\ln(1+x)(1-\cos x) = \frac{1}{2}x^3 - \frac{1}{4}x^4 + o(x^4)$$

■

5.4.3 *SE* of composition of functions

Proposition 5.4.3 — *SE* of composition of functions. Suppose $f(x) \underset{x\to 0}{\longrightarrow} 0$ and $g(u) = a_0 + a_1u + \cdots + a_nu^n + o(u^n)$ when $u \to 0$.
Since we can write $o(u^n) = u^n\varepsilon(u)$ with $\varepsilon \underset{0}{\to} 0$, we have if the composition is allowed

$$g(f(x)) = a_0 + a_1f(x) + \cdots + a_n(f(x))^n + (f(x))^n\varepsilon(f(x))$$

with $\varepsilon(f(x)) \underset{x\to 0}{\longrightarrow} 0$
This can be written as :

$$g(f(x)) = a_0 + a_1f(x) + \cdots + a_n(f(x))^n + o((f(x))^n)$$

We can substitute u by $f(x)$ in the $SE_n(0)$ of $g(u)$ since $f(x) \underset{x\to 0}{\longrightarrow} 0$.
So by knowing a series expansion of f, we can deduce a series expansion of $g(f(x))$.

■ **Example 5.8**

- $SE_6(0)$ of $x \to \ln(1+x^2+x^3)$ when $x \to 0$
  $\ln(1+x^2+x^3) = \ln(1+u)$ with

$$\begin{aligned}
&u = x^2 + x^3 \to 0 \\
&u^2 = x^4 + 2x^5 + x^6 \\
&u^3 = x^6 + o(x^6) \\
&\text{and } o(u^3) = o(x^6)
\end{aligned}$$

  A series expansion to order 3 of $\ln(1+u)$ is enough.

$$\ln(1+u) = u - \frac{1}{2}u^2 + \frac{1}{3}u^3 + o(u^3)$$

  then

$$\ln(1+x^2+x^3) = x^2 + x^3 - \frac{1}{2}x^4 - x^5 - \frac{1}{6}x^6 + o(x^6)$$

- $SE_3(0)$ of $\mathrm{e}^{\frac{1}{1+x}}$, when $x \to 0$

$$\mathrm{e}^{\frac{1}{1+x}} = \mathrm{e}^{1-x+x^2-x^3+o(x^3)} = \mathrm{e}.\mathrm{e}^u$$

  with

$$\begin{aligned}
&u = -x + x^2 - x^3 + o(x^3) \to 0 \\
&u^2 = x^2 - 2x^3 + o(x^3) \\
&u^3 = -x^3 + o(x^3) \\
&\text{and } o(u^3) = o(x^3).
\end{aligned}$$

$$\mathrm{e}^u = 1 + u + \frac{1}{2}u^2 + \frac{1}{6}u^3 + o(u^3)$$

  then

$$\mathrm{e}^{\frac{1}{1+x}} = \mathrm{e} - \mathrm{e}.x + \frac{3\mathrm{e}}{2}x^2 - \frac{13\mathrm{e}}{6}x^3 + o(x^3)$$

■

5.4.4 SE of a Quotient

Proposition 5.4.4 — SE of a Quotient . The regular part of SE of the quotient $\frac{f}{g}$ is the quotient in the division according to the increasing powers of the regular part of f by the regular part of g.

■ **Example 5.9**

- $SE_3(0)$ of $x \to \dfrac{e^x}{1-x}$

$$
\begin{array}{rrrr|l}
1 & +x & +\frac{x^2}{2} & +\frac{x^3}{6} & 1-x \\ \cline{5-5}
-1 & +x & & & \\ \cline{1-3}
 & 2x & +\frac{x^2}{2} & & 1 \quad +2x \quad +\frac{5}{2}x^2+\frac{8}{3}x^3 \\
 & -2x & +2x^2 & & \\ \cline{2-4}
 & & \frac{5}{2}x^2 & +\frac{x^3}{6} & \\
 & & -\frac{5}{2}x^2 & +\frac{5}{2}x^3 & \\ \cline{3-4}
 & & & \frac{8}{3}x^3 &
\end{array}
$$

$$\frac{e^x}{1-x} = 1+2x+\frac{5}{2}x^2+\frac{8}{3}x^3+o(x^3)$$

- $SE_5(0)$ of $x \to \tan x$

$$
\begin{array}{rrr|l}
x & -\frac{1}{6}x^3 & +\frac{1}{120}x^5 & 1- \quad \frac{1}{2}x^2 \quad +\frac{1}{24}x^4 \\ \cline{4-4}
-x & +\frac{x^3}{2} & -\frac{x^5}{24} & \\ \cline{1-3}
 & \frac{x^3}{3} & -\frac{x^5}{30} & x \quad +\frac{1}{3}x^3 \quad +\frac{2}{15}x^5 \\
 & -\frac{x^3}{3} & \frac{x^5}{6} & \\ \cline{2-3}
 & & \frac{2}{15}x^5 &
\end{array}
$$

$$\tan x = x+\frac{1}{3}x^3+\frac{2}{15}x^5+o(x^5).$$

■

5.4.5 *SE* by derivation

Theorem 5.4.5 — *SE* by derivation. Let f be differentiable on I and admits a SE of order n at 0.
If f' admits a SE of order $n-1$ at 0 then the regular part of expansion of f' is the derivative of the regular part of the expansion of f.

■ **Example 5.10** Since $x \to \dfrac{1}{1-x}$ and its derivative $x \to \dfrac{1}{(1-x)^2}$ admit SE of order n and $n-1$ respectively at 0 (as they are of class $\mathscr{C}^\infty$ on $\mathbb{R}^*$), so by differentiating the SE of $x \to \dfrac{1}{1-x}$ we obtain that of its derivative.

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + o(x^n)$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + nx^{n-1} + o(x^{n-1})$$

■

R It is possible that a differentiable function admits a *SE* of order n at 0 without its derivative admitting a *SE* of order $n-1$ at 0. It is necessary to check the conditions of the theorem before using it.

■ **Example 5.11** Let $n \in \mathbb{N}$ and

$$f(x) = \begin{cases} 1 + x + x^2 \cos(1/x), & \text{si } x \neq 0 \\ 1, \quad \text{if } x = 0 \end{cases}$$

When $x \to 0$, we have $f(x) = 1 + x + o(x)$ so then f admits a series expansion of order 1 at 0. On the other hand, its derivative

$$f'(x) = \begin{cases} 1 + 2x\cos(1/x) + \sin(1/x), & \text{if } x \neq 0 \\ 1, \quad \text{if } x = 0 \end{cases}$$

does not admit a limit at 0 and therefore neither a series expansion at 0. ■

5.5 Applications

5.5.1 Determination of equivalents

Definition 5.5.1 — Equivalents of functions. We say that f is equivalent to g at a if we can write at neighborhood of a

$$f(x) = g(x)\theta(x)$$

with $\theta \underset{a}{\to} 1$ We note then $f \sim g$ or $f(x) \sim g(x)$ when $x \to a$.
If the function g does not vanish in the neighborhood of a then we have

$$f(x) \sim g(x) \Leftrightarrow \frac{f(x)}{g(x)} \xrightarrow[x\to a]{} 1$$

■ **Example 5.12**

$$\text{when } x \to +\infty, \quad x^2 + x + 2\ln(x) \sim x^2$$
$$\text{when } x \to 0, \quad x^2 + x + 2\ln(x) \sim 2\ln x$$

■

■ **Example 5.13**

- The first non-zero term of a series expansion provides a simple equivalent of the function studied at the considered point.
- From the series expansions of the usual functions, we obtain these famous equivalents when $x \to 0$:

$$\sin x \sim x,\ \tan x \sim x,\ \ln(1+x) \sim x,\ \mathrm{e}^x - 1 \sim x,\ 1-\cos x \sim \frac{x^2}{2},\ (1+x)^\alpha \sim 1+\alpha x.$$

- Let us determine a simple equivalent of $x^x - x$ when $x \to 1$.
  It is important to begin by changing variable to get near 0.
  Put $x = 1+h$, with $h = x-1 \to 0$

$$x^x - x = (1+h)^{1+h} - (1+h) = \mathrm{e}^{(1+h)\ln(1+h)} - (1+h)$$

but

$$\mathrm{e}^{(1+h)\ln(1+h)} = \mathrm{e}^{(1+h)\left(h-\frac{h^2}{2}+o(h^2)\right)} = \mathrm{e}^{h+\frac{h^2}{2}+o(h^2)} = 1+h+h^2+o(h^2)$$

so

$$x^x - x = h^2 + o(h^2) \sim h^2 \sim (x-1)^2$$

■

5.5.2 Limit determination

Corollary 5.5.1 — *SE* to determine a limit. Obtaining an equivalent makes it possible to obtain the limit of the considered function.

■ **Example 5.14**

- Let's find

$$\lim_{x\to 0} \frac{\tan(2x) - 2\tan x}{\sin(2x) - 2\sin x}$$

when $x \to 0$

$$\tan x = x + \frac{1}{3}x^3 + o(x^3) \quad \text{then} \quad \tan(2x) - 2\tan x \sim 2x^3$$

$$\sin x = x - \frac{1}{6}x^3 + o(x^3) \quad \text{then} \quad \sin(2x) - 2\sin x \sim -x^3$$

Therefore

$$\frac{\tan(2x) - 2\tan x}{\sin(2x) - 2\sin x} \sim \frac{2x^3}{-x^3} \to -2$$

- Let us find

$$\lim_{x\to+\infty} \left(\cos\frac{1}{x}\right)^{x^2}$$

When $x \to +\infty$

$$\left(\cos\frac{1}{x}\right)^{x^2} = \exp\left(x^2 \ln\left(\cos\frac{1}{x}\right)\right)$$

but

$$\cos\frac{1}{x} = 1 - \frac{1}{2x^2} + o\left(\frac{1}{x^2}\right)$$

then $\ln\left(\cos\frac{1}{x}\right) \sim -\frac{1}{2x^2}$ so $x^2\ln\left(\cos\frac{1}{x}\right) \to -\frac{1}{2}$

and finally

$$\lim_{x\to+\infty}\left(\cos\frac{1}{x}\right)^{x^2} = \frac{1}{\sqrt{\mathrm{e}}}$$

■

5.5.3 Local positioning of a curve and its tangent

Corollary 5.5.2 — *SE* to position a curve in relation to its tangent. Let $f : I \to \mathbb{R}$ and $a \in I$.
We assume that f admits a series expansion of order n at a of the form :

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + o\left((x-a)^n\right)$$

The function being defined at a, we have necessarily $a_0 = f(a)$ and $a_1 = f'(a)$.
The equation of the tangent T to f at a is then given by $y = a_0 + a_1(x-a)$.
It suffices then to study the sign of

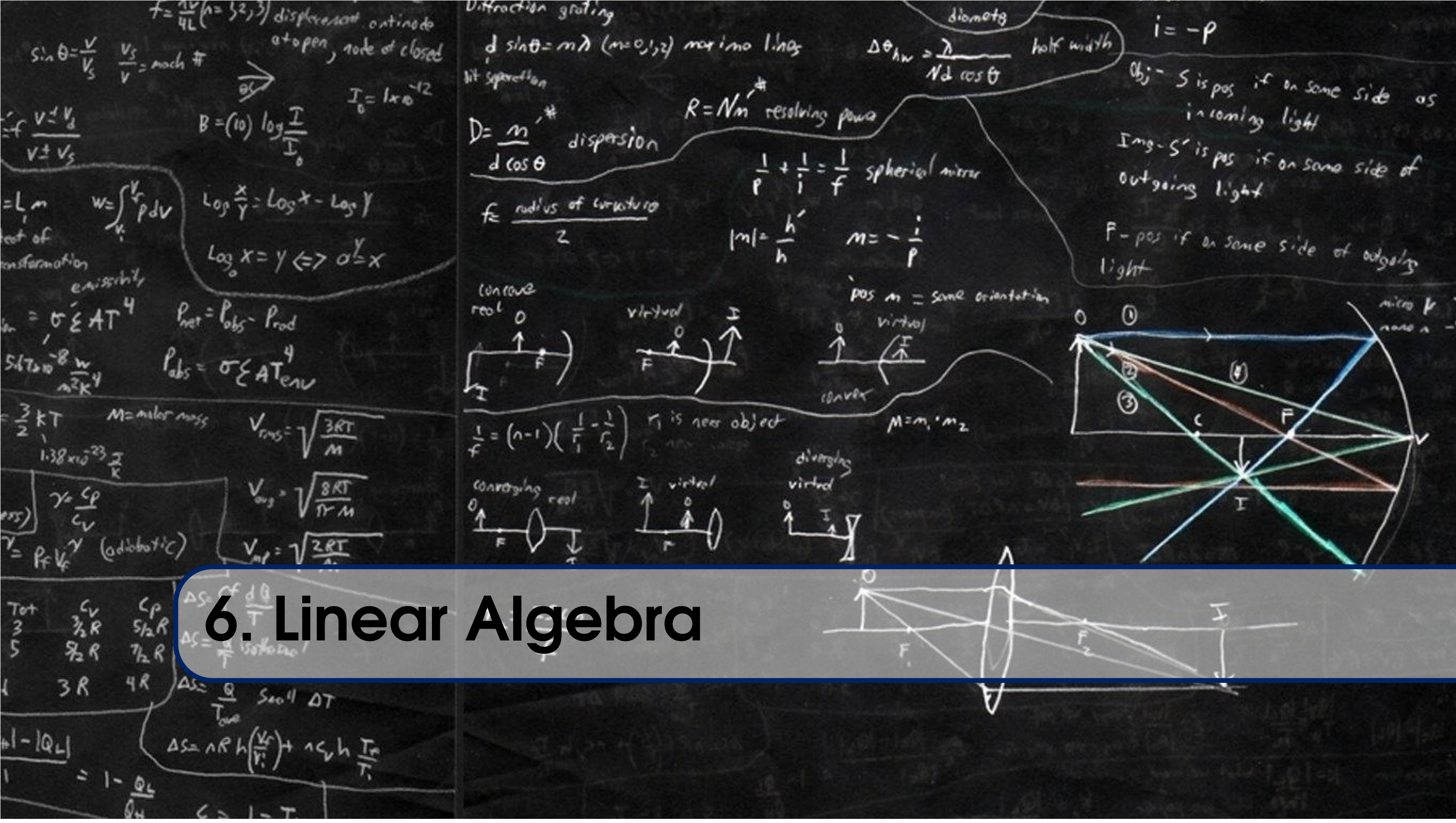
$$f(x) - y = a_2(x-a)^2 + \cdots + a_n(x-a)^n + o\left((x-a)^n\right)$$

to deduce the position of the curve of f with respect to T at a.

■ **Example 5.15** The curve of $x \to \mathrm{e}^x$ is above its tangent at 0 because

$$\mathrm{e}^x - (1+x) \sim \frac{x^2}{2} \geqslant 0, \quad \text{when } x \to 0.$$

■



6. Linear Algebra

6.1 Motivation

The notion of vector space is a fundamental structure of modern mathematics. The aim is to identify the common properties shared by very different sets. For example, we can add two vectors of the plane, and also multiply a vector by a real number (if you want to enlarge or shrink it). But we can also add two functions, or multiply a function by a real number. Same thing with polynomials, matrices, ...
The goal is to obtain general theorems which will apply equally well to vectors of the plane, space, polynomials, matrices,...
In what follows, $\mathbb{K}$ designates a field. In most examples, this will be $\mathbb{R}$: the field of real numbers.

6.2 Internal composition laws

Let E a set.

Definition 6.2.1 — Internal composition laws (ICL). We call **Internal composition law** on E any map from $E \times E$ to E.
When it is appropriate to note $\star$ this law, we note $x \star y$ the image of the pair (x, y) by the previous map.
The element $x \star y$ is called composition of x by y via $\star$.
Internal composition laws are usually denoted $\star, +, \times, \circ, \cdots$

Example 6.1

- On the set $\mathbb{N}$, addition $+$ and multiplication $\times$ are ICLs.
  On the other hand $\div$ is not an ICL on $\mathbb{N}$ since $\dfrac{1}{2} \notin \mathbb{N}$.
- The composition $\circ$ of maps is an ICL on $\mathscr{F}(\mathbb{R}, \mathbb{R})$.

- Union $\cup$ and intersection $\cap$ are ICLs on $\mathscr{P}(E)$.

■

Definition 6.2.2 — External composition laws. We call External composition law (external product) from $\mathbb{K}$ on a set E any map from $\mathbb{K}\times E$ to E :

$$\begin{array}{ccc} \mathbb{K}\times E & \to & E \\ (\lambda, u) & \mapsto & \lambda\cdot u \end{array}$$

An external product is usually denoted by a point.
Elements of $\mathbb{K}$ are called scalars (which often designate real or complex numbers).

■ **Example 6.2**

- On the plane, we define an external product that for each $\lambda\in\mathbb{R}$ and $\vec{v}$ vector associate the vector $\lambda\vec{v}$.
- The reel external product on $\mathscr{F}(\mathbb{R},\mathbb{R})$ that for each $\lambda\in\mathbb{R}$ and $f\in\mathscr{F}(\mathbb{R},\mathbb{R})$ associate the function λf.

■

6.3 Vector space

6.3.1 Definition of a Vector space

Definition 6.3.1 — Vector space. A $\mathbb{K}$ **Vector space** is a non-empty set E provided with :

- an internal composition law, i.e. a map from $E\times E$ to E :

$$\begin{array}{ccc} E\times E & \to & E \\ (u,v) & \mapsto & u+v \end{array}$$

- an external composition law, i.e. a map from $\mathbb{K}\times E$ to E :

$$\begin{array}{ccc} \mathbb{K}\times E & \to & E \\ (\lambda, u) & \mapsto & \lambda\cdot u \end{array}$$

which verify the following properties:

1. $\forall u,v\in E,\ u+v=v+u$ (the law $+$ is Commutative).
2. $\forall u,v,w\in E,\ u+(v+w)=(u+v)+w$ (the law $+$ is Associative).
3. E contains the **Identity element**, i.e.

$$\exists 0_E\in E,\ u+0_E=0_E+u=u, \qquad \text{for all } u \text{ in } E.$$

4. Any element $u\in E$ admits a **symmetrical** u' (noted $-u$), i.e.

$$\forall u\in E,\ \exists u'\in E,\ \text{such that } u+u'=u'+u=0_E$$

5. $1 \cdot u = u \quad$ (for all $u \in E$)
6. $\lambda \cdot (\mu \cdot u) = (\lambda\mu) \cdot u \quad$ (for all $\lambda, \mu \in \mathbb{K}$, $u \in E$)
7. $\lambda \cdot (u + v) = \lambda \cdot u + \lambda \cdot v \quad$ (for all $\lambda \in \mathbb{K}$, $u, v \in E$)
8. $(\lambda + \mu) \cdot u = \lambda \cdot u + \mu \cdot u \quad$ (for all $\lambda, \mu \in \mathbb{K}$, $u \in E$)

Example 6.3

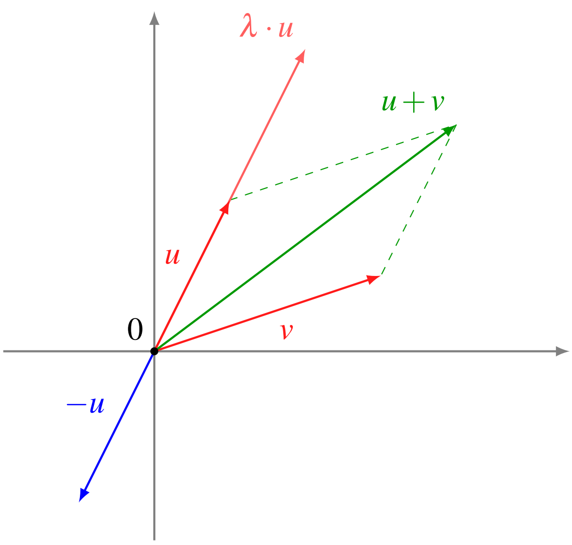
- The $\mathbb{R}$-vector space $\mathbb{R}^2$:
  Put $\mathbb{K} = \mathbb{R}$ and $E = \mathbb{R}^2$. An element $u \in E$ is a pair (x, y) with x element of $\mathbb{R}$ and y element of $\mathbb{R}$. We write $\mathbb{R}^2 = \{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}$.
  - *Internal law verification :* If (x, y) and (x', y') two elements of $\mathbb{R}^2$, then :

$$(x, y) + (x', y') = (x + x', y + y') \in \mathbb{R}^2.$$

  - *External law verification :* If λ is a reel number and (x, y) an element of $\mathbb{R}^2$, then :

$$\lambda \cdot (x, y) = (\lambda x, \lambda y) \in \mathbb{R}^2.$$

  Identity element of the internal law is the null vector $(0, 0)$.
  The symmetrical of (x, y) is $(-x, -y)$, that could also be noted $-(x, y)$.



- Let X a set and $E = \mathscr{F}(X, \mathbb{R})$.
  For $\lambda \in \mathbb{R}$, $f, g : E \to \mathbb{R}$, we define $\lambda f : X \to \mathbb{R}$ and $f + g : X \to \mathbb{R}$ by

$$\forall x \in X, (\lambda.f)(x) = \lambda f(x) \text{ and } (f + g)(x) = f(x) + g(x).$$

  We thus define an exterior product of $\mathbb{R}$ on $\mathscr{F}(X, \mathbb{R})$ and an additive internal composition law on $\mathscr{F}(X, \mathbb{R})$.
  $(\mathscr{F}(X, \mathbb{R}), +, .)$ is an $\mathbb{R}$-vector space whose null vector is the null function.
- $\mathbb{R}$ is an $\mathbb{R}$ vector space. In this situation, scalars and vectors denote elements of $\mathbb{R}$, and the exterior product then corresponds to the multiplication on $\mathbb{R}$.
- The space $(\mathbb{R}_n[X], +, .)$ of polynomials of lower or equal degree to n, is an $\mathbb{R}$-vector space.

However $\left(\{P \in \mathbb{R}_n[X] / P'(0) = 1\}, +, .\right)$ is not a vector space because it does not contain the null polynomial.

■

6.3.2 Terminology and notations

Let's collect the definitions already seen.

- We call the elements of E **vectors**.
- Elements of $\mathbb{K}$ are called **scalars**.
- A $\mathbb{K}$ vector space is a vector space whose scalars are in the field $\mathbb{K}$.
- L' **The identity element** 0_E is also called **the null vector**. It should not be confused with the element 0 of $\mathbb{K}$. When there is no risk of confusion, 0_E will also be noted 0.
- The **symmetrical** $-u$ of a vector $u \in E$ is also called the l'**opposite** of u.
- The ICL on E (usually noted $+$) is called the addition and $u + u'$ is the sum of vectors u and u'.
- The external composition law on E is usually called multiplication by a scalar. The multiplication of a vector u by the scalar λ is usually noted λu, instead of $\lambda \cdot u$.

In a $\mathbb{K}$ vector space we can :

- Add n vectors (defined by induction), $n \geq 2$.

$$v_1 + v_2 + \cdots + v_n = (v_1 + v_2 + \cdots + v_{n-1}) + v_n.$$

  Associativity the law $+$ allows us to get rid of parentheses in the sum $v_1 + v_2 + \cdots + v_n = \sum_{i=1}^{n} v_i$.
- Subtract two vectors u et v i.e. computing $u + (-v)$ noted $u - v$, then $\lambda(u - v) = \lambda u - \lambda v$ and $(\lambda - \mu)u = \lambda u - \mu u$.
- Multiply a vector u by a scalar λ, i.e. computing $\lambda .u$ (and we will avoid writing $u.\lambda$.
- Divide a vector u by a scalar λ, i.e. computing $\frac{1}{\lambda}u$.
- we cannot multiply two vectors, nor divide by a vector because the corresponding laws are not defined.

Proposition 6.3.1 Identity element 0_E and symmetrical elements when they exist are unique. (i.e. one cannot find two different opposites for the same element u).

Proposition 6.3.2 Let E a vector space on a field $\mathbb{K}$. Let $u \in E$ and $\lambda \in \mathbb{K}$.
Then we have :

1. $0 \cdot u = 0_E$
2. $\lambda \cdot 0_E = 0_E$
3. $(-1) \cdot u = -u$
4. $\boxed{\lambda \cdot u = 0_E \iff \lambda = 0 \text{ or } u = 0_E}$

6.3.3 Linear subspace, basic definitions and properties

It is a long task to verify the 8 axioms which make a set a vector space. Fortunately, there is a quick and efficient way to prove that a set is a vector space: thanks to the notion of vector

subspace.
In the following, we will note Vector space and Vector subspace by **E-v** and **S-e-v** respectively.

Definition 6.3.2 — Linear subspace. Let E a $\mathbb{K}$-vector space. A part F of E is called **linear subspace** or **vector subspace** if :

- $0_E \in F$,
- $u+v \in F$ for all $u,v \in F$,
- $\lambda \cdot u \in F$ for all $\lambda \in \mathbb{K}$ and all $u \in F$.

R

- The first condition means that the zero vector of E must also be in F. In fact it is even enough to prove that F is non-empty.
- The second is to say that F is stable for addition: the sum $u+v$ of two vectors u,v of F is of course a vector of E (since E is a vector space), here we require that $u+v$ be an element of F.
- The third condition is to say that F is stable for multiplication by a scalar.

Example 6.4 The set $F = \left\{(x,y) \in \mathbb{R}^2 \mid x+y=0\right\}$ is a subspace of $\mathbb{R}^2$. Indeed :

1. $(0,0) \in F$,
2. if $u=(x_1,y_1)$ and $v=(x_2,y_2)$ belong to F, then $x_1+y_1=0$ and $x_2+y_2=0$ then $(x_1+x_2)+(y_1+y_2)=0$ and so $u+v=(x_1+x_2,y_1+y_2)$ belong to F,
3. if $u=(x,y) \in F$ and $\lambda \in \mathbb{R}$, then $x+y=0$ so $\lambda x+\lambda y=0$, hince $\lambda u \in F$.

However $\left\{(x,y) \in \mathbb{R}^2 \mid x+y=7\right\}$ is not a vector subspace of $\mathbb{R}^2$ because it does not contain the neutral element $(0,0)$. ■

Example 6.5 $A = \{f \in \mathscr{F}(\mathbb{R},\mathbb{R})/f(x) \leqslant 0, \forall x \in \mathbb{R}\}$ is not a vector subspace of $\mathscr{F}(\mathbb{R},\mathbb{R})$ as for $f : x \to x^2 \in A$, we have $-f \notin A$. ■

6.3.4 Vector space and vector subspace

The notion of vector subspace takes on its full interest with the following theorem: a vector subspace is itself a vector space. It is this theorem which will provide us with plenty of examples of vector spaces.

Theorem 6.3.3 Let E be a $\mathbb{K}$-vector space and F be a vector subspace of E. Then F is itself a $\mathbb{K}$-vector space for the laws induced by E.

Linear combination

Definition 6.3.3 — Linear combination. Let $n \geq 1$ an integer, let $v_1,v_2,\dots,v_n$, n vectors of a vector space E. Any vector of the shape

$$u = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$$

(where $\lambda_1,\lambda_2,\dots,\lambda_n$ are elements of $\mathbb{K}$) is called **linear combination** of vectors $v_1,v_2,\dots,v_n$. Les scalaires $\lambda_1,\lambda_2,\dots,\lambda_n$ are called **coefficients** of the linear combination.

R If $n = 1$, then $u = \lambda_1 v_1$, we say that u is **collinear** to v_1.

Proposition 6.3.4 A vector space E contains any linear combination of its vectors.

Example 6.6

- In the $\mathbb{R}$-vector space $\mathbb{R}^3$, $(3,3,1)$ is a linear combination of the vectors $(1,1,0)$ and $(1,1,1)$ because we have equality

$$(3,3,1) = 2(1,1,0) + (1,1,1).$$

- In the $\mathbb{R}$-vector space $\mathbb{R}^2$, the vector $u = (2,1)$ is not collinear with the vector $v_1 = (1,1)$ because if it were, there would exist a real λ such that $u = \lambda v_1$, which is equivalent to $(2,1) = (\lambda, \lambda)$.

■

Theorem 6.3.5 — Characterization of a subspace by the notion of linear combination. Let E a $\mathbb{K}$-vecotr space and F a non-empty part of E.
F is a vector subspace of E if and only if

$$\lambda u + \mu v \in F \qquad \text{for all } u, v \in F \qquad \text{and all } \lambda, \mu \in \mathbb{K}.$$

In other words if and only if any linear combination of two elements of F belongs to F.

Sum of two vector subspaces

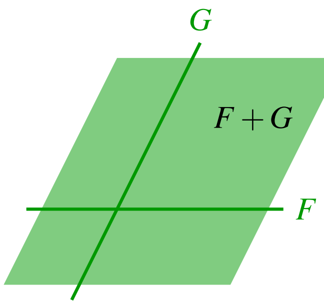
Since the union of two vector subspaces F and G is not in general a vector subspace, it is useful to know vector subspaces which contain both subspaces F and G, and especially the smallest of them (in the sense of inclusion).

Definition 6.3.4 — Sum of two subspaces. Let F and G two vector subspaces of a $\mathbb{K}$-vector space E. The set of all elements $u + v$, where u is an element of F and v an element of G, is called **sum** of vector subspaces F and G. This sum is noted $F + G$. Then we have

$$F + G = \{u + v \mid u \in F, v \in G\}.$$

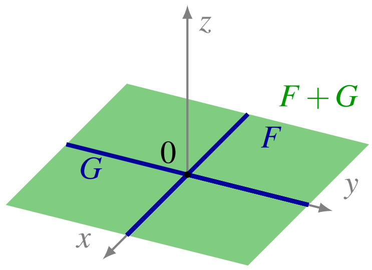
Proposition 6.3.6 Let F and G be two vector subspaces of the $\mathbb{K}$-vector space E.

1. $F + G$ is a vector subspace of E.
2. $F + G$ is the smallest vector subspace containing both F and G.



■ **Example 6.7**

$$F = \{(x,y,z) \in \mathbb{R}^3 \mid y = z = 0\} \quad \text{and} \quad G = \{(x,y,z) \in \mathbb{R}^3 \mid x = z = 0\}.$$



An element w of $F+G$ is written $w = u+v$ where $u \in F$ and $v \in G$. Since $u \in F$ then there exists $x \in \mathbb{R}$ such that $u = (x,0,0)$, and since $v \in G$, there exists $y \in \mathbb{R}$ such that $v = (0,y,0)$. Then $w = (x,y,0)$.
Conversely, such an element $w = (x,y,0)$ is the sum of $(x,0,0)$ and of $(0,y,0)$. So $F+G = \{(x,y,z) \in \mathbb{R}^3 \mid z = 0\}$. We see for this example that any element of $F+G$ is written *uniquely* as the sum of an element of F and an element of G. ■

Direct sum of two subspaces

Definition 6.3.5 Let F and G two vector subspaces of E.
F and G are in "**direct sum**" in E if

- $F \cap G = \{0_E\}$,
- $F+G = E$.

we note $F \oplus G = E$.

Proposition 6.3.7 $F \oplus G = E$ if and only if any element of E is written in a **unique** way as the sum of an element of F and an element of G.

Theorem 6.3.8 — The subspace of all Linear combinations. Let $\{v_1,\dots,v_n\}$ a finite set of vectors of a $\mathbb{K}$-vector space E. So :

- The set of linear combinations of vectors $\{v_1,\dots,v_n\}$ is a vector subspace of E.
- It is the smallest vector subspace of E (in the sense of inclusion) containing the vectors $v_1,\dots,v_n$.

R This vector subspace is called **subspace spanned by** $v_1,\dots,v_n$ and is noted : $\text{span}(v_1,\dots,v_n)$.
We have then

$$u \in \text{span}(v_1,\dots,v_n) \iff \text{there exist } \lambda_1,\dots,\lambda_n \in \mathbb{K} \quad \text{such that}$$

$$u = \lambda_1 v_1 + \cdots + \lambda_n v_n$$

■ **Example 6.8**

- $\text{span}(u) = \{\lambda.u \ / \ \lambda \in \mathbb{K}\}$
- $\text{span}(u,v) = \{\lambda.u + \mu.v \ / \ \lambda,\mu \in \mathbb{K}\}$

- It is very effective to establish that a party is a vector subspace by observing that it is spanned by a set.
  Verify that $F = \{(x,y,z) \in \mathbb{R}^3 \mid x-y-z=0\}$ is a vector subspace of $\mathbb{R}^3$.

  A triple of $\mathbb{R}^3$ is an element of F if and only if $x = y+z$. So u is an element of F if and only if it can be written $u = (y+z,y,z)$. However, we have

$$(y+z,y,z) = y(1,1,0)+z(1,0,1).$$

  So F is the set of linear combinations of $\{(1,1,0),(1,0,1)\}$. So F is the vector subspace $\text{span}\big((1,1,0),(1,0,1)\big)$. It is indeed a vector plane (a plane passing through the origin).

■

6.3.5 Spanning set

Definition 6.3.6 — Spanning set. A family $\mathscr{F} = (e_1,\cdots,e_n)$ of vectors of E is said to be a spanning family of E if any vector of E can be written as a linear combination of elements of $\mathscr{F}$ i.e.

$$\forall x \in E, \exists(\lambda_1,\cdots,\lambda_n) \in \mathbb{K}^n,\ x = \lambda_1 e_1+\cdots+\lambda_n e_n = \sum_{i=1}^{n} \lambda_i e_i$$

R The family $\mathscr{F}$ is spanning of E if and only if, $\text{Vect}\,\mathscr{F} = E$.

■ **Example 6.9**

- In $E = \mathbb{C}$ considered as an $\mathbb{R}$-vector space, the family $\mathscr{F} = (1,i)$ is a spanning family. Indeed, for all $z \in \mathbb{C}$, we can write $z = a.1+b.i$ with $a = \text{Re}(z)$ and $b = \text{Im}(z)$.
- In $E = \mathbb{K}^n$, let $e_i = (0,\cdots,0,1,0,\cdots,0) \in \mathbb{K}^n$ where 1 is in the "ith" position.
  The family $B = (e_1,\cdots e_n)$ is a spanning family of $\mathbb{K}^n$.
  Indeed, for all $x = (x_1,\cdots,x_n) \in \mathbb{K}^n$, we can write $x = x_1e_1+\cdots+x_ne_n$.
- For the vector space $E = \mathscr{F}(\mathbb{R},\mathbb{R})$, there is no spanning family.

■

R In the first example, we notice that $\mathscr{F}_2 = (1,i,2-3i)$ or $\mathscr{F}_2 = (1,i,2-3i,-1+7i)$ also constitute spanning families for $\mathbb{C}$ i.e. any family containing a spanning one, is also a spanning family.

6.3.6 Linear independence

Definition 6.3.7 — Linear independence.

- A family of vectors $(e_1,\cdots,e_n)$ in E is said to be linearly independent if

$$\forall \lambda_1,\cdots,\lambda_n \in \mathbb{K},\ \lambda_1 e_1+\cdots+\lambda_n e_n = 0 \Rightarrow \lambda_1 = \cdots = \lambda_n = 0$$

  We then say that the vectors $e_1,\cdots,e_n$ are linearly independent.

- It is said that the family $(e_1, \cdots, e_n)$ is linearly dependent

$$\exists \lambda_1, \cdots, \lambda_n \in \mathbb{K},\ \lambda_1 e_1 + \cdots + \lambda_n e_n = 0 \text{ and } (\lambda_1, \cdots \lambda_n) \neq (0, \cdots, 0)$$

i.e. the equality $\lambda_1 e_1 + \cdots + \lambda_n e_n = 0$ can be verified with atleast a non zero element from $\lambda_1, \cdots, \lambda_n$.

■ **Example 6.10** Let E a $\mathbb{K}$-v.s and $u \neq 0$ an element of E, the family (u) is linearly independent

$$\forall \lambda \in \mathbb{K}, \lambda u = 0 \Rightarrow \lambda = 0.$$

On the other hand, the family (0) is linearly depenedent. ■

Proposition 6.3.9 Let $n \geqslant 2$ and $(e_1, \cdots e_n)$ a family of vectors of E.
We have the equivalence between :

- $(e_1, \cdots e_n)$ is linearly dependent;
- one of the vectors $e_1, \cdots, e_n$ is a linear combination of the others.

■ **Example 6.11**

- In $\mathbb{C}$ considered as an $\mathbb{R}$ vector space, the family $\mathscr{F} = (1, i, 2-3i)$ is linearly dependent, since $2-3i$ is a linear combination of 1 and i.
- Let $u, v \in E$.
  (u, v) is linearly dependent, if and only if, $u = \lambda v$ for some $\lambda \neq 0$ or $v = \lambda u$ for some $\lambda \neq 0$.
- In $E = \mathbb{R}^3$, consider the vectors $u = (1,2,1)$, $v = (1,-1,1)$, $w = (1,1,0)$ and the family $\mathscr{F} = (u, v, w)$.
  Is this family linearly independent ?
  Let $\alpha, \beta, \gamma \in \mathbb{R}$.

$$\alpha u + \beta v + \gamma w = 0 \Leftrightarrow \begin{cases} \alpha + \beta + \gamma = 0 \\ 2\alpha - \beta + \gamma = 0 \\ \alpha + \beta = 0 \end{cases}$$

After solving the system, we get

$$\alpha u + \beta v + \gamma w = 0 \Leftrightarrow \alpha = \beta = \gamma = 0$$

then, the family $\mathscr{F}$ is linearly independent.

R In case where the system would admit other solutions α, β, γ than the null triple $(0,0,0)$, we conclude that the family is linearly dependent.

■

6.3.7 Basis of a vector space

Definition 6.3.8 — Basis of a vector space. The family $B = (e_1, \cdots, e_n)$ of vectors of E is called a basis of E if it is a spanning linearly independent family of E.

Example 6.12

- In $E = \mathbb{K}^n$, put $e_i = (0, \cdots 0, 1, 0, \cdots, 0) \in \mathbb{K}^n$ where 1 situated in the ith position.
  We have already proved that $B = (e_1, \cdots, e_n)$ is spanning of $\mathbb{K}^n$, let us show that it is also linearly independent.
  Let $\lambda_1, \cdots, \lambda_n \in \mathbb{K}$. Assume $\lambda_1 e_1 + \cdots + \lambda_n e_n = 0$.
  We have $(\lambda_1, \cdots, \lambda_n) = (0, \cdots, 0)$ and then $\lambda_1 = \cdots = \lambda_n = 0$.
  At last, B is linearly independent and spanning of $\mathbb{K}^n$, it is a basis of $\mathbb{K}^n$, called : the canonical basis of $\mathbb{K}^n$.
  Case $n = 1$: $e_1 = 1$, (1) is a basis of $\mathbb{K}$.
  Case $n = 2$: $e_2 = (1,0) = \vec{i}$, $e_2 = (0,1) = \vec{j}$, $(\vec{i}, \vec{j})$ is a basis of $\mathbb{K}^2$.
  Case $n = 3$: $e_1 = (1,0,0) = \vec{i}$, $e_2 = (0,1,0) = \vec{j}$, $e_3 = (0,0,1) = \vec{k}$.
- The family $(1, i)$ is a basis of the $\mathbb{R}$-vector space $\mathbb{C}$.
- The family $(1, X, X^2, \cdots, X^n)$ is a basis of $\mathbb{R}_n[X]$: the vector space of polynomials of lower or equal degree to n.
- The empty set is a basis of $\{0\}$.
- For the vector space $E = \mathscr{F}(\mathbb{R}, \mathbb{R})$, there is no basis.

R a $\mathbb{K}$-vector space E can admit an infinity of bases and they are all made up of the same number of vectors.

■

Theorem 6.3.10 — Components in a basis. If $B = (e_1, \cdots, e_n)$ is a basis of a $\mathbb{K}$-vector space E then

$$\forall x \in E, \exists ! (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{K}^n, x = \lambda_1 e_1 + \cdots \lambda_n e_n$$

The scalars $\lambda_1, \cdots \lambda_n$ are called the components of x in the basis B (or the coordinates of x).

R The components of a vector depend on the base in which we work.
Example : In the $\mathbb{R}$-vector space $\mathbb{C}$, components of $z \in \mathbb{C}$ in the canonical basis $(1, i)$ are $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$.

Definition 6.3.9 — Dimension of a vector space. A $\mathbb{K}$-vector space E is said to be of finite dimension if it has a basis.
We call dimension of E the number of vectors constituting the basis of E. It is noted $\dim E$ or $\dim_{\mathbb{K}} E$.
If the space E is not of finite dimension, we set $\dim E = +\infty$.

Example 6.13

- $\dim \mathbb{K}^n = n$ because the canonical basis of $\mathbb{K}^n$ is formed of n vectors.
- $\dim_{\mathbb{R}} \mathbb{C} = 2$.

- $\dim\{0\} = 0$ because the empty family is the basis of null space.
- Planar geometric visualizations correspond to dimension 2, visualizations in space correspond to dimension 3.

■

6.4 Linear application

6.4.1 Definition of a linear map

Definition 6.4.1 — Linear map. Let E and F two $\mathbb{K}$-vector spaces. A map f from E into F is a **linear map** if it meets the following two conditions:

1. $f(u+v) = f(u) + f(v)$, for all $u, v \in E$;
2. $f(\lambda \cdot u) = \lambda \cdot f(u)$, for all $u \in E$ and all $\lambda \in \mathbb{K}$.

In other words: a map is linear if it «respects » the two laws of a vector space.

R The set of linear applications from E into F is denoted $\mathscr{L}(E,F)$.

■ **Example 6.14** The map f defined by

$$\begin{array}{rccc} f: & \mathbb{R}^3 & \to & \mathbb{R}^2 \\ & (x,y,z) & \mapsto & (-2x, y+3z) \end{array}$$

is linear. Indeed, let $u = (x,y,z)$ and $v = (x',y',z')$ two elements of $\mathbb{R}^3$ and λ a reel number.

$$\begin{aligned} f(u+v) &= f(x+x', y+y', z+z') \\ &= \big(-2(x+x'), y+y'+3(z+z')\big) \\ &= (-2x, y+3z) + (-2x', y'+3z') \\ &= f(u) + f(v) \end{aligned}$$

and

$$\begin{aligned} f(\lambda \cdot u) &= f(\lambda x, \lambda y, \lambda z) \\ &= (-2\lambda x, \lambda y + 3\lambda z) \\ &= \lambda \cdot (-2x, y+3z) \\ &= \lambda \cdot f(u) \end{aligned}$$

■

Proposition 6.4.1 Let E and F two $\mathbb{K}$-vector spaces. If f is a linear map from E into F, then :

- $f(0_E) = 0_F$,
- $f(-u) = -f(u)$, for all $u \in E$.

Proposition 6.4.2 — Characterization of a linear mapping. Let E and F two $\mathbb{K}$-vector spaces and f a map from E into F. The map f is linear if and only if, for all vectors u and v in E and for all scalars λ and μ in $\mathbb{K}$,

$$f(\lambda u+\mu v)=\lambda f(u)+\mu f(v).$$

More generally, a linear map f preserves linear combinations: for all $\lambda_1,\dots,\lambda_n\in\mathbb{K}$ and all $v_1,\dots,v_n\in E$, we have

$$f(\lambda_1 v_1+\cdots+\lambda_n v_n)=\lambda_1 f(v_1)+\cdots+\lambda_n f(v_n).$$

Vocabulary

Let E and F two $\mathbb{K}$-vector spaces.

- A linear map from E into F is also called **homomorphism** of vector space. The set of linear maps from E into F is noted $\mathscr{L}(E,F)$.
- A linear map from E into E is called **endomorphism** of E. The set of endomorphisms of E is noted $\mathscr{L}(E)$.

6.4.2 Image and rank of a linear map

Let E and F two sets and f a map of E into F. Let A a subset of E. The set of images taken by f of all elements of A, is called **direct image** of A by f, and noted $f(A)$. It is a subset of F. We have by definition:

$$f(A)=\{f(x)\mid x\in A\}.$$

Throughout the rest, E and F will designate $\mathbb{K}$-vector spaces and $f:E\to F$ will be a linear map. $f(E)$ is called **image** (or range) of the linear map f and is denoted $\Im f$.

Proposition 6.4.3 — Image structure of a vector subspace.

1. If E' is a vector subspace of E, then $f(E')$ is a vector subspace of F.
2. In particular, $\Im f$ is a vector subspace of F, and its dimension is called rank of f.

R By definition of the direct image $f(E)$:

f is surjective if and only if $\boxed{\Im f=F}$.

6.4.3 Kernel of a linear map

Definition 6.4.2 — Definition of the kernel. Let E and F two $\mathbb{K}$-vector spaces and f a linear map from E into F. The **kernel** of f, noted $\mathrm{Ker}(f)$, is the set of elements of E whose images are 0_F : $\boxed{\mathrm{Ker}(f)=\{x\in E\mid f(x)=0_F\}}$

In other words, the kernel is the reciprocal image of the zero vector of the arrival space: $\mathrm{Ker}(f)=f^{-1}\{0_F\}$.

Proposition 6.4.4 Let E and F two $\mathbb{K}$-vector spaces and f a linear map from E into F. The kernel of f is a vector subspace of E.

■ **Example 6.15** Reconsider the previous map f defined as

$$\begin{array}{rccc} f: & \mathbb{R}^3 & \to & \mathbb{R}^2 \\ & (x,y,z) & \mapsto & (-2x, y+3z) \end{array}$$

Let us find its kernel $\text{Ker}(f)$.

$$\begin{aligned} (x,y,z) \in \text{Ker}(f) &\iff f(x,y,z) = (0,0) \\ &\iff (-2x, y+3z) = (0,0) \\ &\iff \begin{cases} -2x &= 0 \\ y+3z &= 0 \end{cases} \\ &\iff (x,y,z) = (0,-3z,z), \quad z \in \mathbb{R} \end{aligned}$$

So $\text{Ker}(f) = \{(0,-3z,z) \mid z \in \mathbb{R}\}$.
In other words, $\text{Ker}(f) = \text{Vect}\{(0,-3,1)\}$ and $\dim \text{Ker}(f) = 1$. ■

■ **Example 6.16** Let us find the range of f. Let $(x',y') \in \mathbb{R}^2$.

$$\begin{aligned} (x',y') = f(x,y,z) &\iff (-2x, y+3z) = (x',y') \\ &\iff \begin{cases} -2x &= x' \\ y+3z &= y' \end{cases} \end{aligned}$$

We can take for example $x = -\dfrac{x'}{2}$, $y' = y$, $z = 0$.
Conclusion : for any $(x',y') \in \mathbb{R}^2$, we have $f(-\dfrac{x'}{2}, y', 0) = (x',y')$.
So $\Im(f) = \mathbb{R}^2$, $\text{rank}(f) = 2$ and f is surjective. ■

Theorem 6.4.5 — Characterization of injective linear maps. Let E and F two $\mathbb{K}$-vector spaces and f a linear map from E into F. Then :

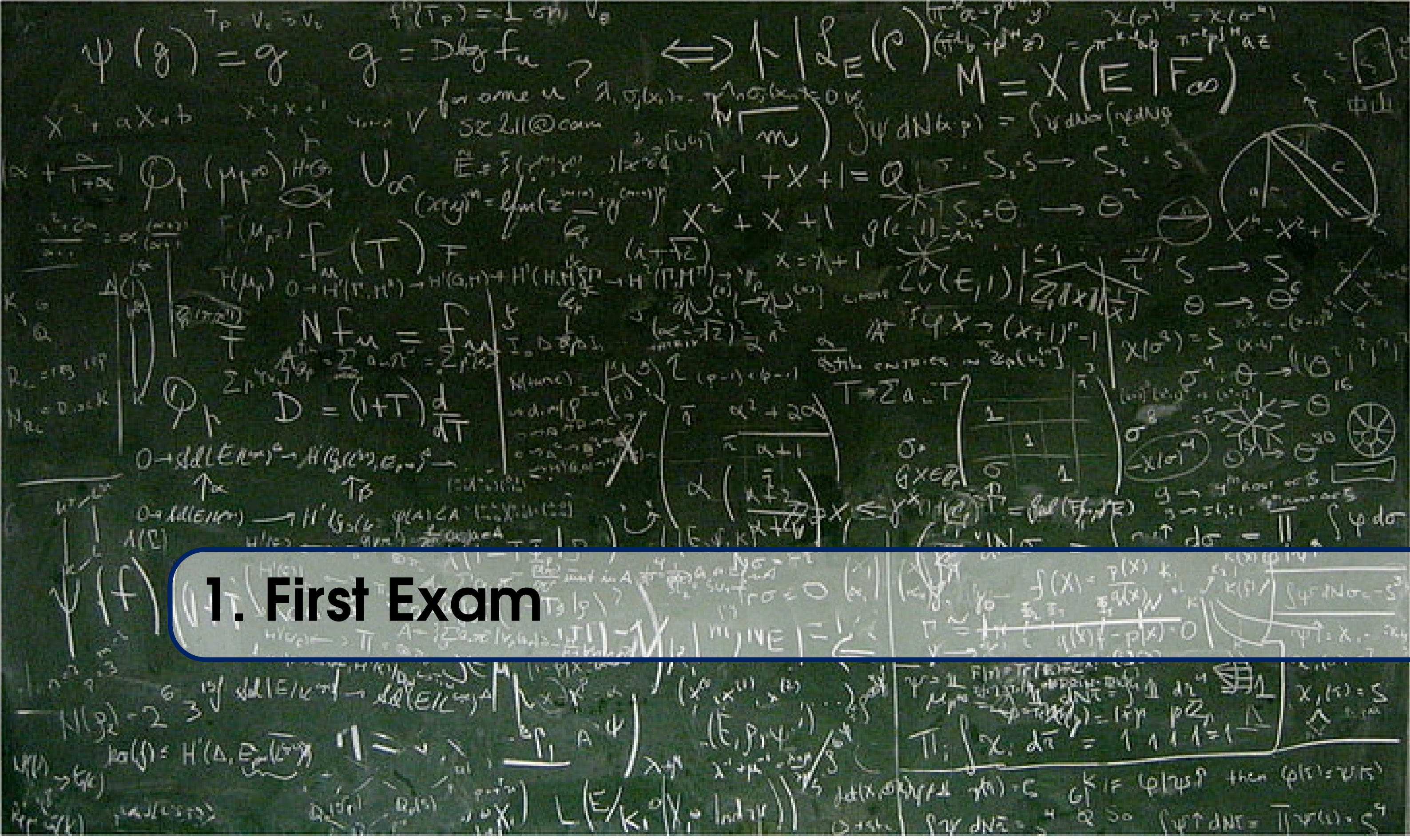
$$\boxed{f \text{ injective} \iff \text{Ker}(f) = \{0_E\}}$$

In other words, f is injective if and only if its kernel contains only the zero vector. In particular, to show that f is injective, it suffices to verify that :

$$\boxed{\text{if } f(x) = 0_F \text{ then } x = 0_E}$$

II

Proposed Exams



1. First Exam

Exercise 1.1 Answer true or false, providing justification:

Let f and g be two functions from $\mathbb{R}$ to $\mathbb{R}$.

1. $fg = 0 \implies f = 0$ or $g = 0$.
2. Every continuous function on $]0,1[$ is bounded.
3. A function can be continuous but not differentiable.
4. Every function is continuous on its domain of definition.
5. Every continuous function is differentiable.
6. Consider the function f defined by:

$$f(x) = \begin{cases} e^x, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0. \end{cases}$$

   Then $f'(0) = 1' = 0$.

7. The function $x \to f(x) = 2 + x - x^2 + x^3 \sin\left(\frac{1}{x}\right)$ has a Taylor series expansion of degree 2 around 0.
8. We know that $(1+x)^\alpha = 1 + \alpha x + o(x)$ as $x \to 0$, so the Taylor expansion of $(1+x)^{\frac{1}{x}}$ is

$$(1+x)^{\frac{1}{x}} = 1 + \frac{1}{x}x + o(x) = 2 + o(x).$$

■

Exercise 1.2 Consider the function f defined by:

$$f(x)=\begin{cases} x^2\sin\left(\frac{1}{x}\right), & \text{if } x\neq 0;\\ 0, & \text{if } x=0.\end{cases}$$

1. Show that f is differentiable on $\mathbb{R}$.
2. Does f have a Taylor series expansion of degree 1 at 0? If yes, provide this expansion.
3. Show that f' is not continuous at 0.
4. Does f have a Taylor series expansion of order 2 at 0? Why or why not?

■

Exercise 1.3 Consider the function f defined by:

$$f(x)=\frac{1}{x}\left(\frac{e^x+e^{-x}}{2}\right).$$

1. What is the domain of definition of f?
2. Examine the parity of f.
3. Find $\lim_{x\to+\infty} f(x)$ and deduce $\lim_{x\to-\infty} f(x)$.
4. Provide the Taylor expansion of f around 0 up to degree 1 ($SE_1(0)$).
5. Show that f can be extended continuously at 0.
6. Show that the equation $f(x)=\frac{1}{2}$ has at least one solution in $]0,+\infty[$.

■

Exercise 1.4 Consider the real function defined by:

$$f(x)=\begin{cases} \frac{3-x^2}{2}, & \text{if } x\in]-\infty,1[;\\ \frac{1}{x}, & \text{if } x\in[1,+\infty[.\end{cases}$$

1. Examine the continuity and differentiability of f at $x_0=1$.
2. Show that f satisfies the hypotheses of the Mean Value Theorem on $[0,2]$.
3. Determine all constants c such that: $f(2)-f(0)=2f'(c)$.

■

Exercise 1.5 Consider the following function:

$$f(x)=e^x\ln(1+x).$$

1. Show that f establishes a bijection from $]-1,+\infty[$ to $\mathbb{R}$.
2. Provide the Taylor series expansion up to degree 4 around 0 for f.
3. Using Taylor-Young's formula, find the value of $f^{(4)}(0)$.

4. Provide the equation of the tangent line at $x = 0$, specifying its position relative to the curve at the point 0.
5. Use Taylor series expansions to calculate the following limit:

$$\lim_{x \to 0} \frac{e^{2x} \ln^2(1+x) - x^2 - x^3}{x^4}.$$

Solutions

Solution 1.1 Answer true or false, providing justification:
Let f and g be two functions from $\mathbb{R}$ to $\mathbb{R}$.

1. $fg = 0 \implies f = 0$ or $g = 0$.
   False. Counterexample: $f(x) = x$ if $x > 0$ and $f(x) = 0$ otherwise, $g(x) = 0$ for $x < 0$ and $g(x) = x$ otherwise.
2. Every continuous function on $]0,1[$ is bounded.
   False.
3. A function can be continuous but not differentiable.
   True. Example: $f(x) = |x|$.
4. Every function is continuous on its domain of definition.
   False. Counterexample: $f(x) = 1$ for $x \neq 0$ and $f(0) = 2$.
5. Every continuous function is differentiable.
   False. Example: $f(x) = |x|$ is continuous but not differentiable at $x = 0$.
6. Consider the function f defined by:

$$f(x) = \begin{cases} e^x, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0. \end{cases}$$

   Then $f'(0) = 1' = 0$.
   False. $f'(0) = 1$.
7. The function $x \mapsto f(x) = 2 + x - x^2 + x^3 \sin\left(\frac{1}{x}\right)$ has a Taylor series expansion of degree 2 around 0.
   True. The expansion is $2 + x - x^2 + O(x^3)$.
8. We know that $(1+x)^\alpha = 1 + \alpha x + o(x)$ as $x \to 0$, so the Taylor expansion of $(1+x)^{\frac{1}{x}}$ is

$$(1+x)^{\frac{1}{x}} = 1 + \frac{1}{x}x + o(x) = 2 + o(x).$$

   False. $(1+x)^{1/x} \to e$ as $x \to 0$.

Solution 1.2

1. **Differentiability of f on $\mathbb{R}$:** To check differentiability at $x \neq 0$, we compute the derivative using the product rule:

$$f'(x) = \left(x^2 \sin\left(\frac{1}{x}\right)\right)' = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right).$$

For $x = 0$, we check the definition of the derivative:

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} = \lim_{h\to 0} h\sin\left(\frac{1}{h}\right).$$

Since $\left|\sin\left(\frac{1}{h}\right)\right| \leq 1$, we have:

$$\left|h\sin\left(\frac{1}{h}\right)\right| \leq |h| \to 0 \quad \text{as } h \to 0.$$

Thus,

$$\lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 0 \quad \Rightarrow \quad f'(0) = 0.$$

Hence, f is differentiable on $\mathbb{R}$ and:

$$f'(x) = \begin{cases} 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

2. **Taylor series expansion of degree 1 at 0:** Since f is differentiable at 0 and $f(0) = 0$, $f'(0) = 0$, the Taylor series expansion of f at 0 up to degree 1 is:

$$f(x) \approx f(0) + f'(0)x = 0 + 0 \cdot x = 0.$$

Therefore, the Taylor series expansion of degree 1 at 0 is simply $f(x) \approx 0$.

3. **f' is not continuous at 0:** To check the continuity of f' at 0, we need to evaluate $\lim_{x\to 0} f'(x)$:

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \left(2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right).$$

Note that $\cos\left(\frac{1}{x}\right)$ oscillates between -1 and 1 as $x \to 0$, hence this limit does not exist. Therefore, f' is not continuous at 0.

4. **Taylor series expansion of order 2 at 0:** Since f' is not continuous at 0, f is not C^1 at 0. Therefore, f cannot have a Taylor series expansion of order 2 at 0.

Solution 1.3

1. **Domain of definition:** The function $f(x)$ involves a division by x. Therefore, the domain of definition is:
$$\mathscr{D}_f = \mathbb{R} \setminus \{0\}.$$
2. **Parity of f:** To check the parity, we compute $f(-x)$:
$$f(-x) = \frac{1}{-x}\left(\frac{e^{-x}+e^{x}}{2}\right) = -\frac{1}{x}\left(\frac{e^{x}+e^{-x}}{2}\right) = -f(x).$$
Since $f(-x) = -f(x)$, the function f is odd.
3. **Limits at $\pm\infty$:**
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{1}{x}\left(\frac{e^{x}+e^{-x}}{2}\right) = \lim_{x\to+\infty} \frac{e^{x}}{2x} = +\infty.$$
Using the parity of f, we deduce:
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} -f(-x) = -\lim_{x\to+\infty} f(x) = -\infty.$$
4. **Taylor expansion around 0 up to degree 1 ($\mathbf{SE}_1(0)$):** First, we expand e^x and e^{-x} around 0:
$$e^x = 1 + x + \frac{x^2}{2} + o(x^2), \quad e^{-x} = 1 - x + \frac{x^2}{2} + o(x^2).$$
Therefore,
$$\frac{e^x+e^{-x}}{2} = \frac{(1+x+\frac{x^2}{2}+o(x^2)) + (1-x+\frac{x^2}{2}+o(x^2))}{2} = 1 + \frac{x^2}{2} + o(x^2).$$
Thus,
$$f(x) = \frac{1}{x}\left(1 + \frac{x^2}{2} + o(x^2)\right) = \frac{1}{x} + \frac{x}{2} + o(x).$$
The Taylor expansion of f around 0 up to degree 1 is:
$$f(x) \approx \frac{1}{x} + \frac{x}{2}.$$
5. **Continuous extension at 0:** To show f can be extended continuously at 0, we need to find $\lim_{x\to 0} f(x)$.
$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x}\left(\frac{e^{x}+e^{-x}}{2}\right).$$
Using the Taylor expansion, we get:
$$\lim_{x\to 0}\left(\frac{1}{x} + \frac{x}{2}\right) = \text{undefined}.$$

However, since $\dfrac{e^x+e^{-x}}{2} \approx 1+\dfrac{x^2}{2}$, we need to redefine f at 0:

$$f(0)=1.$$

Thus, f can be extended continuously at 0 by defining $f(0)=1$.

6. **Existence of solutions to $f(x)=\frac{1}{2}$ in $]0,+\infty[$:** Consider the equation $f(x)=\frac{1}{2}$:

$$\frac{1}{x}\left(\frac{e^x+e^{-x}}{2}\right)=\frac{1}{2}.$$

$$\frac{e^x+e^{-x}}{x}=1.$$

For $x\in]0,+\infty[$, we see that $f(x)$ is continuous and tends to $+\infty$ as $x\to 0$ and as $x\to+\infty$. By the Intermediate Value Theorem, there exists at least one $c\in]0,+\infty[$ such that $f(c)=\frac{1}{2}$.

Solution 1.4

1. Continuity and differentiability at $x_0=1$:
   - **Continuity:** We need to check if $\lim_{x\to 1^-} f(x)=f(1)=\lim_{x\to 1^+} f(x)$.

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\frac{3-x^2}{2}=\frac{3-1^2}{2}=1.$$

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\frac{1}{x}=\frac{1}{1}=1.$$

   Thus, $f(1)=1$, and f is continuous at $x=1$.
   - **Differentiability:** We need to check if $\lim_{h\to 0}\dfrac{f(1+h)-f(1)}{h}$ exists.

$$f'(1^-)=\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{\frac{3-(1+h)^2}{2}-1}{h}=\lim_{h\to 0^-}\frac{-h}{h}=-1.$$

$$f'(1^+)=\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{\frac{1}{1+h}-1}{h}==\lim_{h\to 0^+}\frac{-1}{1+h}=-1.$$

   Since $f'(1^-)=f'(1^+)$, f is differentiable at $x=1$ and $f'(1)=-1$.
2. Mean Value Theorem (MVT) on $[0,2]$:
   - f is continuous on $[0,2]$.
   - f is differentiable on $(0,2)$.

   Therefore, f satisfies the hypotheses of the MVT on $[0,2]$.
3. Constants c such that $f(2)-f(0)=2f'(c)$:

- Calculate $f(2)$ and $f(0)$:

$$f(2) = \frac{1}{2}, \quad f(0) = \frac{3-0^2}{2} = \frac{3}{2}.$$

- Therefore,

$$f(2) - f(0) = \frac{1}{2} - \frac{3}{2} = -1.$$

- We need to find $c \in (0,2)$ such that:

$$-1 = 2f'(c).$$

- For $x \in [0,1)$, $f'(x) = -x$. For $x \in (1,2]$, $f'(x) = -\frac{1}{x^2}$.
- Solve for c:

$$2(-c) = -1 \implies c = \frac{1}{2}, \quad c \in [0,1).$$

Thus, $c = \frac{1}{2}$ is the constant that satisfies the equation.

Solution 1.5

1. The function f is differentiable and its derivative is given by:

$$f'(x) = e^x \ln(1+x) + \frac{e^x}{1+x} > 0, \quad \forall x > -1,$$

this implies in particular that f is strictly increasing. Moreover,

$$\lim_{x \to -1^+} f(x) = -\infty, \qquad \lim_{x \to +\infty} f(x) = +\infty.$$

Since f is also continuous, f realizes a bijection from $]-1,+\infty[$ to $\mathbb{R}$.

2. We have:

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + o\left(x^4\right),$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o\left(x^4\right),$$

and thus

$$f(x) = \mathrm{e}^x \ln(1+x) = x + \frac{x^2}{2} + \frac{x^3}{3} + o\left(x^4\right).$$

3. The Taylor series formula allows us to write,

$$f(x) = \mathrm{e}^x \ln(1+x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + o\left(x^4\right).$$

From the previous result, we immediately have: $f^{(4)}(0) = 0$.

4. The Taylor expansion of f gives the equation of the tangent at $x = 0$, which is $y = x$.

We have

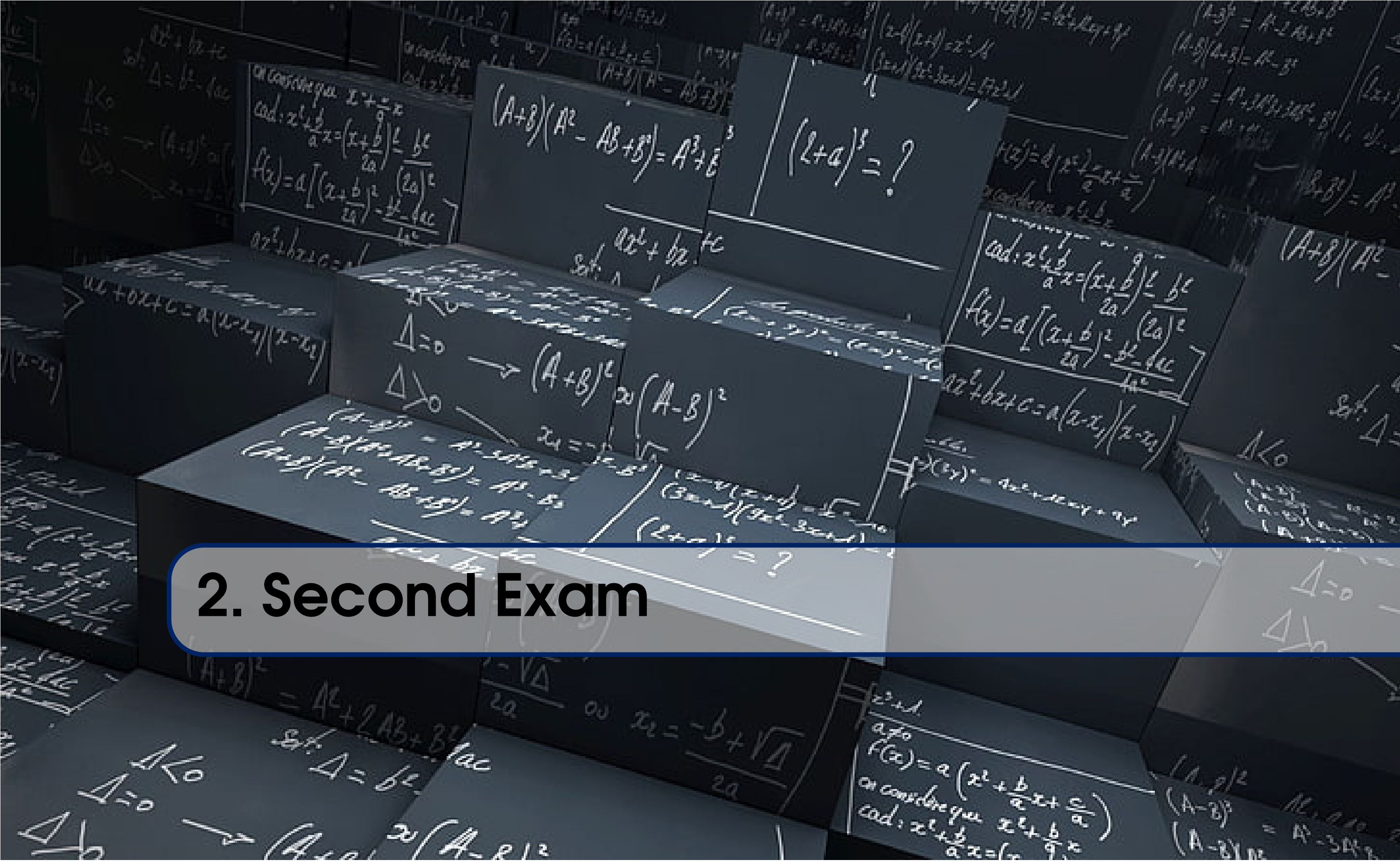
$$f(x) - x = \frac{x^2}{2} + \frac{x^3}{3} + o\left(x^4\right),$$

$\frac{x^2}{2}$ is always positive, so the curve is above its tangent near 0.

5. Calculation of the limit.

$$\begin{aligned} e^{2x} \ln^2(x+1) = (f(x))^2 &= \left(x + \frac{x^2}{2} + \frac{x^3}{3}\right)^2 + o\left(x^4\right) \\ &= x^2 + x^3 + \frac{11}{12}x^4 + o\left(x^4\right) \end{aligned}$$

$$\lim_{x \to 0} \frac{e^{2x} \ln^2(1+x) - x^2 - x^3}{x^4} = \lim_{x \to 0} \frac{\frac{11}{12}x^4 + o\left(x^4\right)}{x^4} = \frac{11}{12} + \lim_{x \to 0} \frac{o\left(x^4\right)}{x^4} = \frac{11}{12}.$$



2. Second Exam

Exercise 2.1 Let $f : E \to F$ $(E, F \subset \mathbb{R})$ be a function defined by:

$$f(x) = \frac{x+1}{2x-3}.$$

1. Provide the conditions on E and F for f to become a bijective function.
2. Determine its inverse function f^{-1}.

■

Exercise 2.2 Let $\mathscr{R}$ be a binary relation defined on $\mathbb{R}^*$ by:

$$\forall x, y \in \mathbb{R}^*, \quad x\mathscr{R}y \quad \Leftrightarrow \quad y\left(x^2+1\right) = x\left(y^2+1\right)$$

1. Show that $\mathscr{R}$ is an equivalence relation.
2. Determine the equivalence class of 1.

■

Exercise 2.3

1. Let f be the function defined by:

$$f(x) = \begin{cases} \dfrac{1-\sqrt{1-x^2}}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(a) Show that the function f is continuous on $[-1,1]$.
(b) Calculate the derivative $f'(x)$ for $x \neq 0$.
(c) Provide the Taylor expansion around $x_0 = 0$ up to order $n = 4$ for the function $\sqrt{1-x^2}$.
(d) Using the previous question, analyze the differentiability of f at the point 0.
(e) Can the Mean Value Theorem be applied to the function f on the interval $[-1,1]$? Justify your answer.

2. Let $h(x) = \text{Arcsin}\left(\dfrac{1-\sqrt{1-x^2}}{x}\right)$.
(a) Calculate $h'(x)$ for $x \neq 0$.
(b) Deduce the Taylor expansion around $x_0 = 0$ up to order $n = 3$ for the function $h(x)$.
(c) Compute $\lim\limits_{x\to 0} \dfrac{h(x) - \frac{x}{2}}{x^3}$.
(d) Find the equation of the tangent to the curve $g(x) = \text{Arctan}\left(\dfrac{1}{x}\right)$ at the point $(1, g(1))$.

■

Solutions

Solution 2.1 Let the function $f(x) = \dfrac{x+1}{2x-3}$. For f to be defined, we must have:

$$2x-3 \neq 0 \Leftrightarrow x \neq \frac{3}{2} \implies E = \mathbb{R} \setminus \left\{\frac{3}{2}\right\}.$$

1. **Injectivity:** $\forall x_1, x_2 \in \mathbb{R} \setminus \{\frac{3}{2}\}, f(x_1) = f(x_2) \overset{?}{\Rightarrow} x_1 = x_2$.

$$\begin{aligned} f(x_1) = f(x_2) &\Rightarrow \frac{x_1+1}{2x_1-3} = \frac{x_2+1}{2x_2-3} \\ &\Rightarrow (x_1+1)(2x_2-3) = (x_2+1)(2x_1-3) \\ &\Rightarrow 2x_1x_2 + 2x_2 - 3x_1 - 3 = 2x_1x_2 + 2x_1 - 3x_2 - 3 \\ &\Rightarrow 5x_2 = 5x_1 \\ &\Rightarrow x_1 = x_2 \\ &\Rightarrow f \text{ is injective.} \end{aligned}$$

To find F, let's verify if f is surjective.

Surjectivity: $\forall y \in F, \exists x \in \mathbb{R} \setminus \{\frac{3}{2}\}, f(x) = y.$

$$\begin{aligned} y = \frac{x+1}{2x-3} &\Leftrightarrow (2x-3)y = x+1 \\ &\Leftrightarrow 2xy - xy = 3y+1 \\ &\Leftrightarrow x(2y-1) = 3y+1 \\ &\Leftrightarrow x = \frac{3y+1}{2y-1}, \forall y \in \mathbb{R} \setminus \{\frac{1}{2}\}. \end{aligned}$$

Observe:

$$x = \frac{3}{2} \implies \frac{3y+1}{2y-1} = \frac{3}{2} \implies 6y+2 = 6y-3 \implies 2 = -3 \text{ (which is impossible)}.$$

Therefore, $x \in \mathbb{R} \setminus \{\frac{3}{2}\}$. We have shown that f is surjective.

Conclusion: f is bijective from $E = \mathbb{R} \setminus \{\frac{3}{2}\}$ to $F = \mathbb{R} \setminus \{\frac{1}{2}\}$.

2. **Inverse function:**

$$\begin{aligned} f^{-1} : \mathbb{R} \setminus \{\tfrac{1}{2}\} &\longrightarrow \mathbb{R} \setminus \{\tfrac{3}{2}\} \\ y &\longmapsto f^{-1}(y) = \frac{3y+1}{2y-1}. \end{aligned}$$

Solution 2.2 On $\mathbb{R}^*$, we have:

$$\forall x, y \in \mathbb{R}^*, \quad x\mathscr{R}y \quad \Leftrightarrow \quad y(x^2+1) = x(y^2+1).$$

1. Let's show that $\mathscr{R}$ is an equivalence relation.

   Reflexivity: $\forall x \in \mathbb{R}^*, x\mathscr{R}x?$

$$\begin{aligned} x\mathscr{R}x &\Leftrightarrow x(x^2+1) = x(x^2+1) \\ &\Leftrightarrow x^3 + x = x^3 + x \\ &\Leftrightarrow x^3 + x = x^3 + x \\ &\Leftrightarrow x^3 - x^3 = x - x \\ &\Leftrightarrow 0 = 0. \end{aligned}$$

   Symmetry: $\forall x, y \in \mathbb{R}^*, x\mathscr{R}y \overset{?}{\Leftrightarrow} y\mathscr{R}x.$

$$\begin{aligned} x\mathscr{R}y &\Leftrightarrow y(x^2+1) = x(y^2+1) \\ &\Leftrightarrow x(y^2+1) = y(x^2+1) \\ &\Leftrightarrow y\mathscr{R}x. \end{aligned}$$

Transitivity: $\forall x, y, z \in \mathbb{R}^*, (x\mathscr{R}y)$ and $(y\mathscr{R}z) \Rightarrow x\mathscr{R}z$.

$$\begin{cases} x\mathscr{R}y \\ \text{and} \\ y\mathscr{R}z \end{cases} \Leftrightarrow \begin{cases} y(x^2+1) = x(y^2+1) \Leftrightarrow y^2+1 = \dfrac{y(x^2+1)}{x} \cdots (\star) \\ \text{and} \\ z(y^2+1) = y(z^2+1) \cdots (\star\star) \end{cases}$$

Substituting $(\star)$ into $(\star\star)$:

$$z \cdot \frac{y(x^2+1)}{x} = y(z^2+1) \Rightarrow z(x^2+1) = x(z^2+1) \Leftrightarrow x\mathscr{R}z.$$

Hence, $\mathscr{R}$ is an equivalence relation.

2. Equivalence class: $\mathrm{Cl}(1) = \{y \in \mathbb{R}^* / y\mathscr{R}1\} = \{1\}$,
because

$$y\mathscr{R}1 \Leftrightarrow 1 \cdot (y^2+1) = y(1^2+1) \Leftrightarrow y^2+1 = 2y \Leftrightarrow y^2 - 2y + 1 = 0.$$

We have a double root $x_1 = x_2 = 1$ (with $\Delta = 0$).

Solution 2.3 Let the function f be defined by:

$$f(x) = \begin{cases} \dfrac{1-\sqrt{1-x^2}}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

I. (a) Continuity of f: $f(0) = 0$.
f is continuous on $[-1, 0[\cup]0, 1]$ because it is a composition of continuous functions on $[-1, 0[\cup]0, 1]$.
For $x = 0$:

$$\lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x} = \lim_{x\to 0} \frac{1-(1-x^2)}{x\left(1+\sqrt{1-x^2}\right)} = \lim_{x\to 0} \frac{x}{1+\sqrt{1-x^2}} = 0 = f(0),$$

$$\Rightarrow f \text{ is continuous at } 0,$$

$$\Rightarrow f \text{ is continuous on } [-1, 1].$$

(b) Calculation of $f'(x)$ for $x \neq 0$:

$$f'(x) = \left(\frac{1-\sqrt{1-x^2}}{x}\right)' = \frac{1}{\sqrt{1-x^2}} - \frac{1-\sqrt{1-x^2}}{x^2}, \quad \forall x \neq 0.$$

(c) The Taylor series expansion of $\sqrt{1-x^2}$ up to order 4:

$$\sqrt{1-x^2} = (1+\underbrace{(-x^2)}_{X})^{1/2}, \quad \text{with } (1+X)^{1/2} = 1+\frac{X}{2}-\frac{X^2}{8}+o(X^2)$$
$$= 1-\frac{x^2}{2}-\frac{x^4}{8}+o(x^4).$$

(d) Differentiability of f: f is differentiable on $[-1,0[\cup]0,1]$ as a composition of differentiable functions.

For $x=0$:

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{f(x)}{x}$$
$$= \lim_{x\to 0}\frac{\frac{1-\left(1-\frac{x^2}{2}-\frac{x^4}{8}\right)}{x}}{x}$$
$$= \lim_{x\to 0}\frac{1}{2}+\frac{x^2}{8} = \frac{1}{2} = f'(0).$$

Hence, f is differentiable at 0 and thus differentiable on $[-1,1]$.

(e) To apply the Mean Value Theorem, the function must be continuous on $[-1,1]$, differentiable on $]-1,1[$, and $f(-1)=f(1)$.

From previous questions, f is continuous on $[-1,1]$, differentiable on $]-1,1[$. It remains to check if $f(-1)=f(1)$?

$$f(-1) = -1 \neq 1 = f(1),$$

therefore, the conclusion is that we cannot apply the Mean Value Theorem.

II. Let the function be:

$$h(x) = \text{Arcsin}\left(\frac{1-\sqrt{1-x^2}}{x}\right).$$

(a) The derivative of $h(x)$:

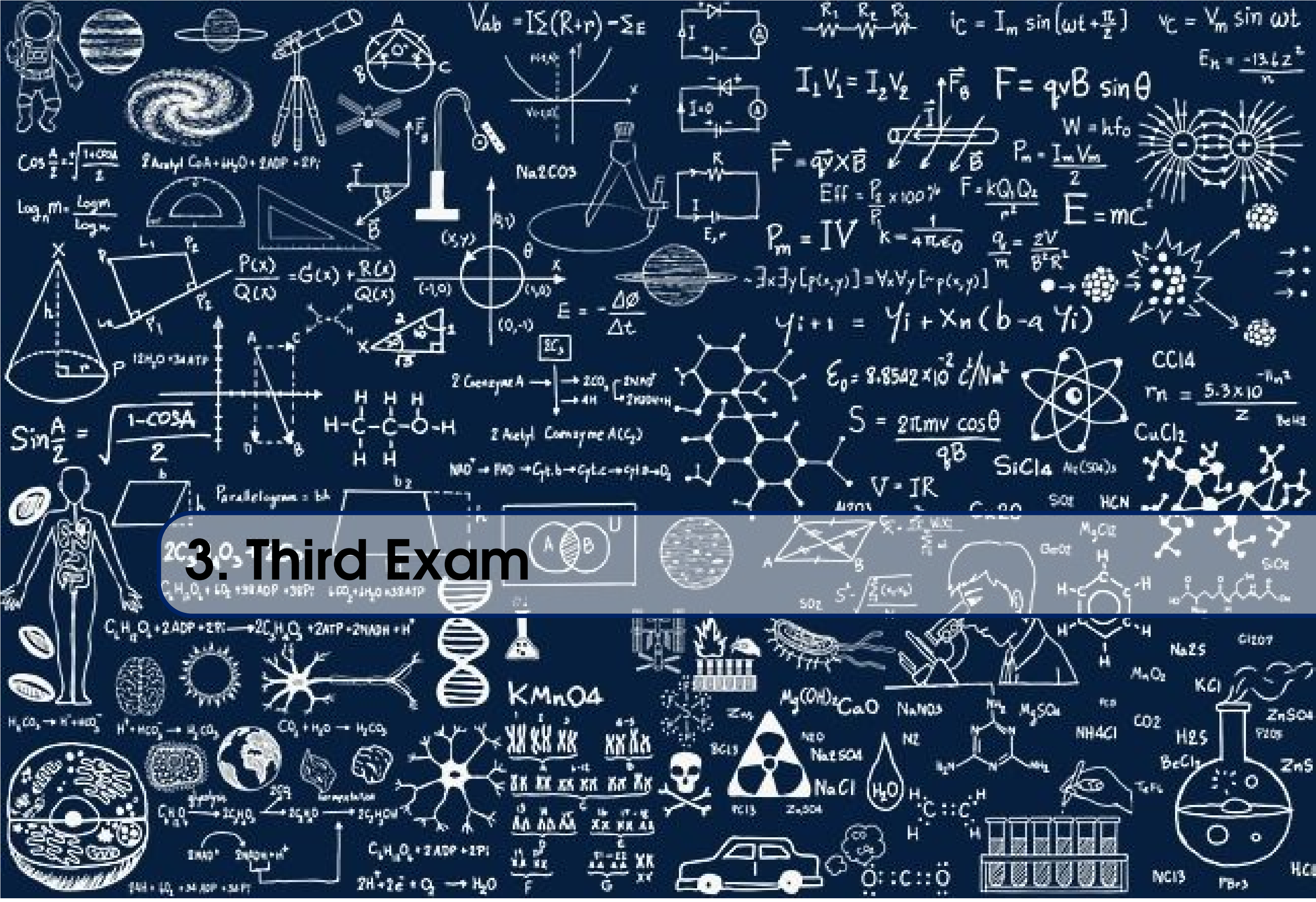
$$h'(x) = \frac{\frac{1}{\sqrt{1-x^2}}-\frac{1-\sqrt{1-x^2}}{x^2}}{\sqrt{1-\left(\frac{1-\sqrt{1-x^2}}{x}\right)^2}}.$$

(b) The Taylor series expansion of $h(x)$ up to order 3:

$$\arcsin\left(\frac{1-\sqrt{1-x^2}}{x}\right) = \arcsin\left(\frac{x}{2}+\frac{x^3}{8}\right) = \frac{x}{2}+\frac{7x^3}{48}+o(x^3).$$

(c) Calculation of $\lim_{x\to 0} \frac{h(x)-\frac{x}{2}}{x^3}$:

$$\lim_{x\to 0} \frac{h(x)-\frac{x}{2}}{x^3} = \frac{\frac{x}{2}+\frac{7x^3}{48}-\frac{x}{2}}{x^3} = \frac{7}{48}.$$



3. Third Exam

Exercise 3.1

1. Answer true or false:
   (a) Every function is a mapping.
   (b) A function can be continuous but not differentiable.
   (c) If a function is twice differentiable at a point, it is of class C^2.
2. State the Mean Value Theorem.
3. Provide the expression of the Taylor-Young formula for a function $f : [a,b] \to \mathbb{R}$, n times differentiable at a point $x_0 \in [a,b]$.

■

Exercise 3.2 Let:

$$\begin{aligned} f: \quad \mathbb{R} &\to \mathbb{R} \\ x &\mapsto f(x) = \frac{2x}{1+x^2}. \end{aligned}$$

1. Is the function f injective?
2. Show that $f(\mathbb{R}) = [-1,1]$; is f surjective?
3. Show that the function g defined by:

$$\begin{aligned} g: \quad [-1,1] &\to [-1,1] \\ x &\mapsto g(x) = f(x), \end{aligned}$$

is a bijection, and determine its inverse. ■

Exercise 3.3 Consider the following function:

$$f(x) = e^x \ln(1+x).$$

1. Show that f establishes a bijection from $]-1,+\infty[$ to $\mathbb{R}$.
2. Provide the Taylor series expansion up to order 4, around 0 for f.
3. Using Taylor-Young's formula, find the value of $f^{(4)}(0)$.
4. Provide the equation of the tangent at $x = 0$, specifying its position relative to the curve at the point 0.
5. Use Taylor series expansions to calculate the following limit:

$$\lim_{x\to 0} \frac{e^{2x}\ln^2(1+x) - x^2 - x^3}{x^4}.$$

■

Solutions

Solution 3.1

1. Answer true or false:
   (a) True. Every function is indeed a mapping, as it assigns exactly one output to each input.
   (b) True. A function can be continuous but not differentiable. A classic example is the absolute value function $f(x) = |x|$, which is continuous everywhere but not differentiable at $x = 0$.
   (c) False. If a function is twice differentiable at a point, it means its second derivative exists at that point. However, for a function to be of class C^2, it must not only have a second derivative but also that the derivative itself has to be differentiable. Thus, being twice differentiable does not necessarily imply that the function is of class C^2.
2. The Mean Value Theorem states: If f is continuous on $[a,b]$ and differentiable on (a,b), then there exists at least one point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

3. The Taylor-Young formula for a function $f : [a,b] \to \mathbb{R}$, n times differentiable at a point $x_0 \in [a,b]$, is given by:

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + R_n(x),$$

where $f^{(k)}(x_0)$ denotes the k-th derivative of f evaluated at x_0, and the remainder term $R_n(x)$ satisfies

$$R_n(x) = \frac{f^{(n)}(c)}{n!}(x - x_0)^n,$$

for some c between x_0 and x.

Solution 3.2 1. **Injectivity of f:**

- To determine if f is injective, we check if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
- Calculate $f(x)$:

$$f(x) = \frac{2x}{1+x^2}$$

- Assume $f(x_1) = f(x_2)$:

$$\frac{2x_1}{1+x_1^2} = \frac{2x_2}{1+x_2^2}$$

- Cross-multiplying gives:

$$2x_1(1+x_2^2) = 2x_2(1+x_1^2)$$

$$2x_1 + 2x_1x_2^2 = 2x_2 + 2x_2x_1^2$$

$$2x_1 - 2x_2 = 2x_2x_1^2 - 2x_1x_2^2$$

$$2(x_1 - x_2) = 2x_1x_2(x_1 - x_2)$$

- If $x_1 \neq x_2$, divide by $2(x_1 - x_2)$:

$$1 = x_1x_2$$

- Therefore, $x_1 = x_2$ or $x_1x_2 = 1$. Thus, f is not injective.

2. **Surjectivity of f:**
   - To show $f(\mathbb{R}) = [-1,1]$, we need to demonstrate that for every $y \in [-1,1]$, there exists an $x \in \mathbb{R}$ such that $f(x) = y$.
   - Analyzing the function $f(x) = \dfrac{2x}{1+x^2}$:

$$\lim_{x \to \pm\infty} f(x) = 0$$

$$f(0) = 0$$

$$f(1) = \frac{2 \cdot 1}{1+1^2} = 1, \quad f(-1) = \frac{2 \cdot (-1)}{1+(-1)^2} = -1$$

   - As $x \to \pm\infty$, $f(x)$ approaches 0, and by the Intermediate Value Theorem, f takes every value in $[-1,1]$.
   - Therefore, $f(\mathbb{R}) = [-1,1]$, indicating that f is not surjective.

3. **Bijection of g:**

- Define $g(x) = f(x)$ for $x \in [-1, 1]$.
- To show g is a bijection, we verify injectivity and surjectivity:
  - **Injectivity**: Already shown.
  - **Surjectivity**: Since $f(\mathbb{R}) = [-1, 1]$, g is surjective onto $[-1, 1]$.
  - Therefore, g is a bijection on $[-1, 1]$.
- **Inverse of** g:

$$g^{-1}(y) = f^{-1}(y) = \frac{1}{2}\left(y - \frac{1-y^2}{1+y^2}\right)$$

Solution 3.3

1. The function f is differentiable and its derivative is given by:

$$f'(x) = e^x \ln(1+x) + \frac{e^x}{1+x} > 0, \quad \forall x > -1,$$

this implies in particular that f is strictly increasing. Moreover,

$$\lim_{x \to -1^+} f(x) = -\infty, \qquad \lim_{x \to +\infty} f(x) = +\infty.$$

Since f is also continuous, f realizes a bijection from $]-1, +\infty[$ to $\mathbb{R}$.

2. We have:

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + o\left(x^4\right),$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o\left(x^4\right),$$

and thus

$$f(x) = \mathrm{e}^x \ln(1+x) = x + \frac{x^2}{2} + \frac{x^3}{3} + o\left(x^4\right).$$

3. The Taylor series formula allows us to write,

$$f(x) = \mathrm{e}^x \ln(1+x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + o\left(x^4\right).$$

From the previous result, we immediately have: $f^{(4)}(0) = 0$.

4. The Taylor expansion of f gives the equation of the tangent at $x = 0$, which is $y = x$. We have

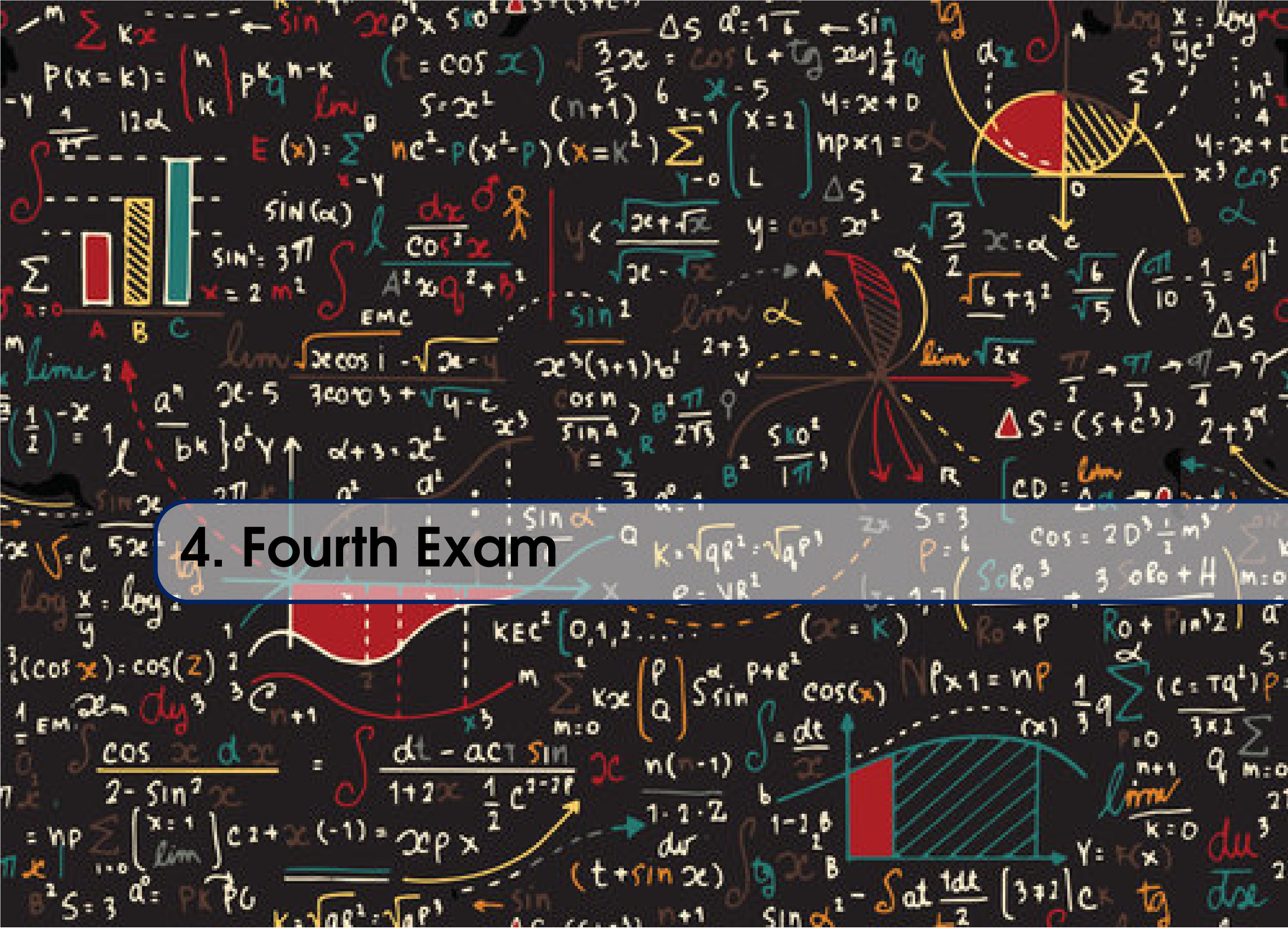
$$f(x) - x = \frac{x^2}{2} + \frac{x^3}{3} + o\left(x^4\right),$$

$\frac{x^2}{2}$ is always positive, so the curve is above its tangent near 0.

5. Calculation of the limit.

$$\begin{aligned} e^{2x}\ln^2(x+1) = (f(x))^2 &= \left(x+\frac{x^2}{2}+\frac{x^3}{3}\right)^2 + o\left(x^4\right) \\ &= x^2+x^3+\frac{11}{12}x^4+o\left(x^4\right) \end{aligned}$$

$$\lim_{x\to 0}\frac{e^{2x}\ln^2(1+x)-x^2-x^3}{x^4} = \lim_{x\to 0}\frac{\frac{11}{12}x^4+o\left(x^4\right)}{x^4} = \frac{11}{12}+\lim_{x\to 0}\frac{o\left(x^4\right)}{x^4} = \frac{11}{12}.$$



4. Fourth Exam

Exercise 4.1 Let's equip the set $E = \mathbb{R}^2$ with the relation $\mathscr{R}$ defined by

$$(x,y)\mathscr{R}(x',y') \iff \exists a > 0, \exists b > 0 \mid x' = ax \text{ and } y' = by.$$

1. Show that $\mathscr{R}$ is an equivalence relation.
2. Determine the equivalence class of $(1,0)$ and $(0,-1)$.

■

Exercise 4.2 Consider the function $f : \mathbb{R} - \{-2\} \longrightarrow \mathbb{R} - \{1\}, x \longmapsto f(x) = \dfrac{x-1}{x+2}$.

1. Show that the function f is a bijection.
2. Calculate the inverse function f^{-1}.
3. Determine $f \circ f^{-1}$.

■

Exercise 4.3 **I)** Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by: $f(x) = \dfrac{\sin x}{x^2 + x}$.

(a) Determine its domain of definition D_f.
(b) Study the continuity of f at the point 0.
(c) Show that the function f can be continuously extended at the point 0 and specify this extension.

(d) Calculate $f'(x)$.

II) Consider the function h defined by: $h(x) = \begin{cases} \dfrac{\sin x}{x^2+x} & \text{if } -1 < x < 0 \\ \dfrac{1+\ln(1+x)}{x+1} & \text{if } x \geqslant 0 \end{cases}$.

(a) State the Intermediate Value Theorem (I.V.T.).

(b) If h is continuous on $\left[-\frac{\pi}{4}, 1\right]$, can we apply I.V.T. to the function h on $\left[-\frac{\pi}{4}, 1\right]$. ■

Exercise 4.4 1. Using a $SE_4(0)$, calculate:

$$\lim_{x \to 0} \frac{e^{x^2} - 1}{\cos x - 1}.$$

2. Calculate the $SE_4(0)$ of : $\dfrac{1+x+x^2}{1-x+x^2}$. ■

Solutions

Solution 4.1 1. We verify the 3 properties of an equivalence relation:

- $\mathscr{R}$ is reflexive: indeed, we have $x = 1 \cdot x$ and $y = 1 \cdot y$.
- $\mathscr{R}$ is symmetric: if $(x,y)\mathscr{R}(x',y')$, then there exist $a,b > 0$ such that $x' = ax$ and $y' = by$. But then, $x = \frac{1}{a}x'$ and $y = \frac{1}{b}y'$, with $\frac{1}{a} > 0$ and $\frac{1}{b} > 0$, so $(x',y')\mathscr{R}(x,y)$.
- $\mathscr{R}$ is transitive: if $(x,y)\mathscr{R}(x',y')$ and if $(x',y')\mathscr{R}(x'',y'')$, then there exist $a,b,c,d > 0$ such that $x' = ax, y' = by, x'' = cx'$ and $y'' = dy'$. But then, $x'' = (ac)x$ and $y'' = (bd)y$ with $ac > 0$ and $bd > 0$. We deduce that $(x,y)\mathscr{R}(x'',y'')$.

Conclusion: $\mathscr{R}$ is an equivalence relation.

2. We have $(x,y)\mathscr{R}(1,0)$ if and only if $(1,0)\mathscr{R}(x,y)$ if and only if there exist $a > 0$ and $b > 0$ such that $x = a \times 1$ and $y = b \times 0 = 0$. Thus, the equivalence class of $(1,0)$ is $]0,+\infty[\times\{0\}$. Similarly, it can be shown that the equivalence class of $(0,-1)$ is $\{0\}\times]-\infty,0[$.

Solution 4.2 Consider the function $f : \mathbb{R}-\{-2\} \longrightarrow \mathbb{R}-\{1\}, x \longmapsto f(x) = \dfrac{x-1}{x+2}$.

1. To show that f is a bijection, we need to demonstrate injectivity and surjectivity.
   - Injectivity: Suppose $f(x_1) = f(x_2)$ for some $x_1, x_2 \in \mathbb{R}-\{-2\}$. Then, solve the equation $\dfrac{x_1-1}{x_1+2} = \dfrac{x_2-1}{x_2+2}$ to show $x_1 = x_2$.
   - Surjectivity: Given any $y \in \mathbb{R}-\{1\}$, find $x \in \mathbb{R}-\{-2\}$ such that $f(x) = y$.

   Therefore, f is a bijection.

2. Let $y = \frac{x-1}{x+2}$. Solve for x to find $f^{-1}(y)$. One finds : $f^{-1}(x) = \frac{1+2x}{1-x}$.
3. Show that $f \circ f^{-1} = \mathrm{Id}$.

Solution 4.3

I) Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by: $f(x) = \frac{\sin x}{x^2+x}$.

(a) $D_f = \mathbb{R} - \{0, -1\}$.

(b) At $x = 0$, the function f is not defined, thus not continous at 0.

(c) Observe that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin(x)}{x} \frac{1}{x+1} = 1.$$

The extension is :

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

(d) One finds :

$$f'(x) = \frac{\cos(x)(x^2+x) - (2x+1)\sin(x)}{(x^2+x)^2}.$$

II) Consider the function h defined by: $h(x) = \begin{cases} \dfrac{\sin x}{x^2+x} & \text{if } -1 < x < 0 \\ \dfrac{1+\ln(1+x)}{x+1} & \text{if } x \geqslant 0 \end{cases}$.

(a) The Intermediate Value Theorem (I.V.T.) states that if a function f is continuous on a closed interval $[a,b]$, then for any value c between $f(a)$ and $f(b)$, there exists at least one $x \in [a,b]$ such that $f(x) = c$.

(b) Yes, we can apply the Intermediate Value Theorem to h on the interval $\left[-\frac{\pi}{4}, 1\right]$ if h is continuous on this interval. The conditions of the I.V.T. are satisfied when the function is continuous on the closed interval.

Solution 4.4

1. The Taylor series expansion for e^x and $\cos x$ around $x = 0$ are given by:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Substitute the TSE expressions into the limit:

$$\lim_{x\to 0} \frac{e^{x^2} - 1}{\cos x - 1} = \lim_{x\to 0} \frac{\left(1 + x^2 + \frac{x^4}{2!} + \dots\right) - 1}{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) - 1}$$

Simplify the expression by canceling out common terms:

$$\lim_{x\to 0} \frac{x^2 + \frac{x^4}{2!} + \dots}{-\frac{x^2}{2!} + \frac{x^4}{4!} - \dots}$$

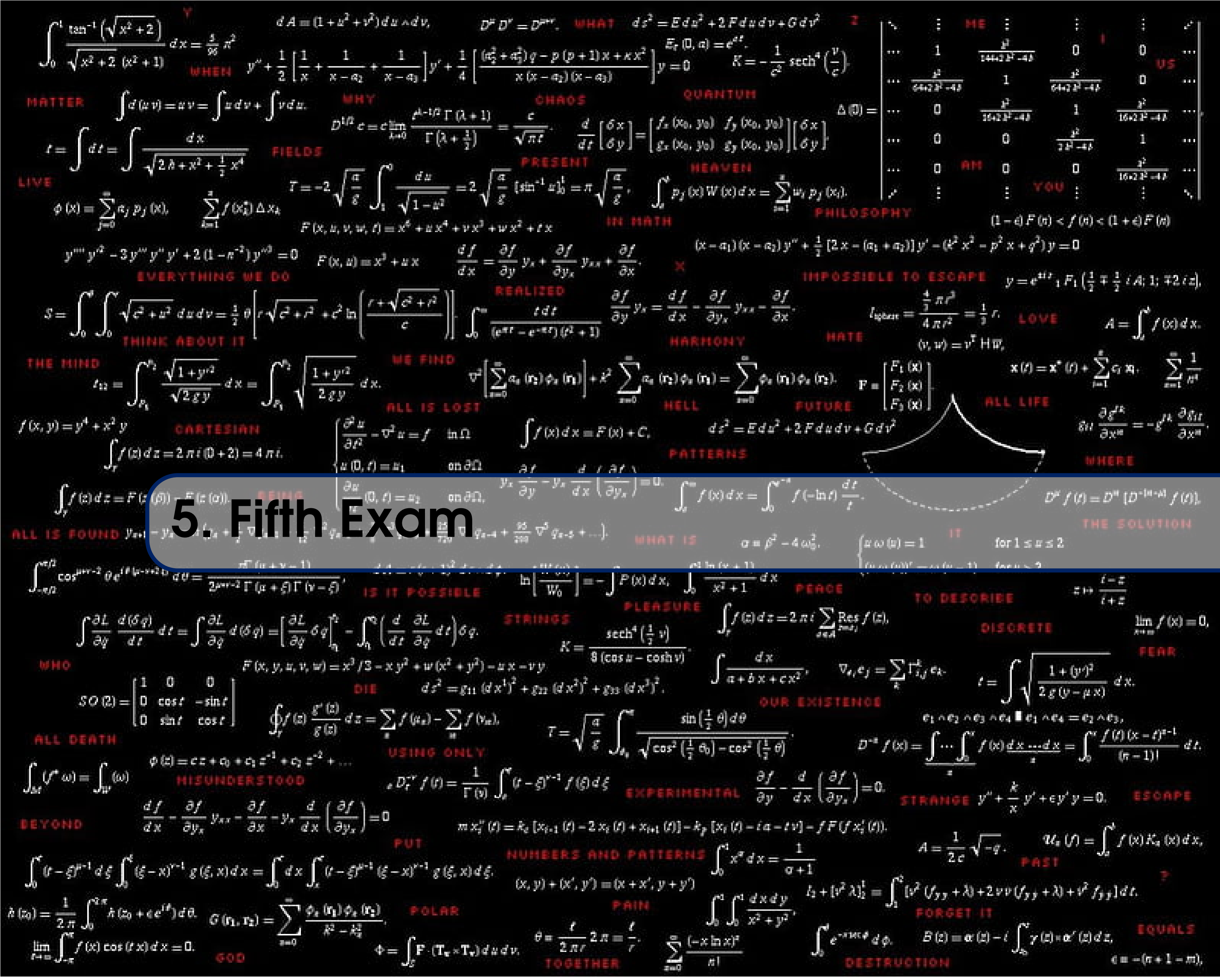
Taking the limit as x approaches 0, higher-order terms become negligible:

$$\lim_{x\to 0} \frac{x^2}{-\frac{x^2}{2!}} = -2$$

Therefore, $\lim_{x\to 0} \frac{e^{x^2} - 1}{\cos x - 1} = -2$.

2. Calculate the $SE_4(0)$ of:

$$\frac{1+x+x^2}{1-x+x^2} = 1 + 2x + 2x^2 - 2x^4 + o(x^4).$$



5. Fifth Exam

Exercise 5.1 Let $f : \mathbb{R} - \{1\} \to \mathbb{R}$ be the function defined by $f(x) = \dfrac{1}{x^2 - 2x + 1}$.

1. Is it injective? Surjective?
2. Let $g :]1; +\infty[\to \mathbb{R}$ be the function defined by $g(x) = f(x)$. Is it injective? Surjective?
3. Let $h : [2; +\infty[\to [0; 1]$ be the function defined by $h(x) = f(x)$. Is it injective? Surjective?

■

Exercise 5.2 On the set $E = \mathbb{R}$, we define the binary relation $\mathscr{T}$ as follows:

$$\forall a, b \in \mathbb{R}, a \mathscr{T} b \iff ab \geq a + b$$

1. Is this relation reflexive? Symmetric? Antisymmetric? Transitive?
2. The same questions on $E = [2, +\infty[$.

■

Exercise 5.3 At altitude x, the gravitational field strength of the Earth is described by the function

$$\begin{array}{rcl} g: \mathbb{R}^+ & \to & \mathbb{R}^+ \\ x & \mapsto & \dfrac{GM}{(R+x)^2}, \end{array}$$

where G is the gravitational constant, M is the mass of the Earth, and R is the radius of the Earth.

- For R sufficiently large compared to x, provide a Taylor expansion ($SE_2(0)$) of g.

Exercise 5.4 Let $f : E \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x^3 \cos\left(\dfrac{1}{x}\right) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ x^3 \sin\left(\dfrac{1}{x}\right) & \text{if } x < 0. \end{cases}$$

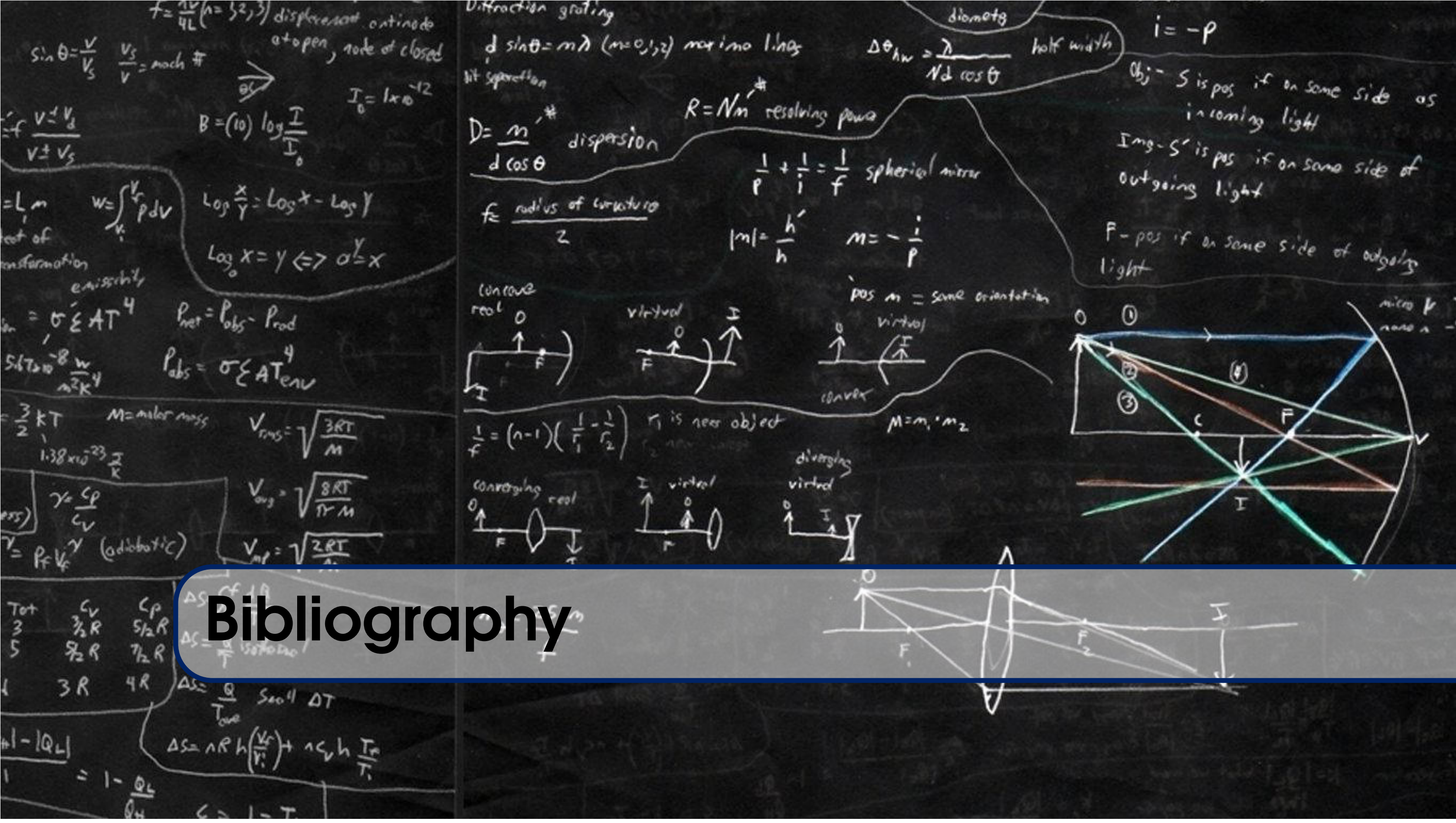
1. Determine E the domain of definition of f.
2. Establish if f is continuous.
3. Calculate $f'(x)$ for $x \neq 0$. Deduce the equation of the tangent line to f at $x = \dfrac{1}{\pi}$.
4. Establish if f is differentiable.

Solutions

Solution 5.1

1. **For the function** $f(x) = \dfrac{1}{x^2 - 2x + 1}$**:**
    - **Injectivity and Surjectivity:**
        - The function $f(x)$ is defined as $f(x) = \dfrac{1}{(x-1)^2}$.
        - $f(x)$ is not injective.
        - $f(x)$ is not surjective since it does not cover all real numbers.
2. **For the function** $g(x) = f(x)$ **on** $]1; +\infty[$**:**
    - **Injectivity and Surjectivity:**
        - $g(x)$ is injective.
        - $g(x)$ is not surjective since no covering of all positive real numbers is done.
3. **For the function** $h(x) = f(x)$ **on** $[2; +\infty[$**:**
    - **Injectivity and Surjectivity:**
        - $h(x)$ is injective.
        - $h(x)$ is surjective.

R The author let the last 3 exercises to the readers interested in practicing what they learned.



Bibliography

[1] Ventre, A. G. S. (2023). *Calculus and Linear Algebra*. Springer International Publishing.

[2] Anton, H., & Kaul, A. (2019). *Elementary Linear Algebra, 12th Edition*. Wiley.

[3] Hass, J., Heil, C., & Weir, M. (2017). *Thomas' Calculus: Early Transcendentals, Single Variable*. Pearson.

[4] Lay, David C., McDonald, Judi J., & Lay, Steven R. (2018). *Linear Algebra and Its Applications*.

[5] Thomas, G. B., & Finney, R. L. (1995). *Calculus and Analytic Geometry (9th Edition)*.

[6] Arnaud Bodin, Marc Bourdon, Sophie Chemla (L'équipe Exo7), *ALGEBRE. COURS DE MATHÉMATIQUES PREMIÈRE ANNÉE*, Université de Lille 1, Version 1.00 – January 2016.

[7] Arnaud Bodin, Léa Blanc-Centi, Niels Borne, Benjamin Boutin, Laura Desideri, Pascal Romon, *ANALYSE. COURS DE MATHÉMATIQUES PREMIÈRE ANNÉE*, Université de Lille 1, Version 1.00 – January 2016.

[8] ATTAR Ahmed, MIRI Sofiane Elhadi, *Algèbre et Analyse Recueil d'Exercices Corrigés*, Faculté des Sciences, Département de Mathématiques, Université Abou Bekr Belkaid Tlemcen, March 8, 2018.

[9] Bernard Ycart, *Développements limités*, Université Joseph Fourier, Grenoble.

[10] L. Brandolese, M-A. Dronne, *ALGEBRE LINÉAIRE Cours et exercices*, ISPB, Faculté de Pharmacie de Lyon, Filière ingénieur, 2014 - 2015.

[11] Bruno Vallette, *L'ALGÈBRE LINÉAIRE POUR TOUS*, Laboratoire J.A.Dieudonné, Université de Nice Sophia-Antipolis, October 8, 2015.

[12] David Delaunay, *Cours mathématiques MPSI*, http://mp.cpgedupuydelome.fr, August 26, 2013.

[13] David Delaunay, *Algèbre linéaire 1 : Résumé de cours*, Université de Bordeaux, Licence de Sciences, Technologies, Santé Mathématiques, Informatique, Sciences de la Matière et Ingénierie, April 25, 2017.

[14] Cl. Gabriel, *Mathématiques avancées*, http://ww3.ac-poitiers.fr/math/mol/mole.htm, 2015-2016.

[15] HITTA Amara, *Cours Algèbre et Analyse I et Exercices Corrigés*, Université 8 Mai 1945 - Guelma.

[16] B. Landreau, D. Schaub, *Notes de Cours d'ALGEBRE LINEAIRE*, Département de Mathématiques, Université d'Angers.

[17] Stéphane Balac, Frédéric Sturm, *Algebre et analyse. Cours de mathématiques de première année avec exercices corrigés*, Collection des sciences appliquées de l'INSA de Lyon, July 2003.

[18] Xavier Dussau, Jean Esterle, Fouad Zarouf, Rachid Zarouf, *ESTIA 1e Année-Mathématiques Cours d'algèbre linéaire Edition 2008*, Collection des sciences appliquées de l'INSA de Lyon, November 26, 2008.